A NOTE ABOUT A CERTAIN GROUPOID C^*-ALGEBRA MAP

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Abstract. We prove that if H is an open subgroupoid of G then the map which extends continuous compactly supported functions on H as identically zero outside H implements an isometric $*$-homomorphism of the reduced twisted groupoid C^*-algebras corresponding to H and G.

We recall that a locally compact groupoid is said to be étale if the range and source maps are local homeomorphisms. An étale groupoid is always r-discrete (i.e. the unit space is an open set) and admits a left Haar system which is essentially given by counting measures [4].

We denote by $\mathbb{T}$ the complex unit circle.

Proposition 1. *Let G be a locally compact Hausdorff étale groupoid endowed with the counting Haar system and let $\sigma \in Z^2(G, \mathbb{T})$. Let H be an open subgroupoid of G endowed with the restricted Haar system, such that $H^{(0)} = G^{(0)}$. Let $j : C_c(H, \sigma) \to C_c(G, \sigma)$ be the map given by extending functions as identically zero outside H. Then j extends to an isometric $*$-homomorphism $j : C_r^*(H, \sigma) \to C_r^*(G, \sigma)$.*

Proof. It is clear that the map j is well-defined: let $f \in C_c(H)$ with $K = \operatorname{supp} f$ compact. The set K is closed in G since G is Hausdorff, and f is zero on ∂K. Since H is open, the extension to G is continuous and compactly supported.

It is straightforward to check that j is linear and $*$-preserving. We now show that j is multiplicative. Let $f, g \in C_c(H)$. Since $\operatorname{supp} j(f) \subseteq H$ and $\operatorname{supp} j(g) \subseteq H$, it follows that $\operatorname{supp}(j(f) * j(g)) \subseteq H$ because H is closed under the groupoid operations of G. Thus it suffices to verify that $[j(f) * j(g)](x) = [j(f * g)](x)$ for all $x \in H$. Fix $x \in H$.

$$\begin{aligned}
[j(f) * j(g)](x) &= \int_G j(f)(xy) j(g)(y^{-1}) \sigma(xy, y^{-1}) d\lambda^{d(x)}(y) \\
&= \int_H j(f)(xy) j(g)(y^{-1}) \sigma(xy, y^{-1}) d\lambda^{d(x)}(y) \\
&= \int_H f(xy) g(y^{-1}) \sigma(xy, y^{-1}) d\lambda^{d(x)}(y) \\
&= (f * g)(x) \\
&= [j(f * g)](x)
\end{aligned}$$

where we used the fact that the Haar system of H is the restriction of the Haar system of G.

We prove next that j is an isometry. Let δ be a probability measure on $H^{(0)}$ concentrated on v (we will abuse notation slightly and denote also by δ the probability measure on $G^{(0)}$ supported on v). By definition, $\operatorname{Ind}_\delta^G(j(f))$ acts on $L^2(G, \nu^{-1})$, where in this case $\nu = \int \lambda^u d\delta(u) = \lambda^v$, hence ν^{-1} is counting measure on G_v and $L^2(\nu^{-1}) \simeq \ell^2(G_v)$. Furthermore, we can write for $\psi \in \ell^2(G_v)$,

$$[\operatorname{Ind}_\delta^G(j(f))\psi](x) = \sum_{y \in G^v} j(f)(xy)\psi(y^{-1})\sigma(xy, y^{-1})$$

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The operator $\mathrm{Ind}_\delta^H(f)$ acts analogously on $\ell^2(H_v)$.

Let H act on the left of G as a subgroupoid and let $\mathcal{O}$ denote the set of orbits of elements of G_v under this action. For each $\zeta \in \mathcal{O}$ we define the subspaces

$$M_\zeta = \{\psi \in \ell^2(G_v) : \psi(x) = 0 \text{ if } x \notin \zeta\}.$$

Observe that

$$\ell^2(G_v) = \bigoplus_{\zeta \in \mathcal{O}} M_\zeta.$$

Moreover, for each $\zeta \in \mathcal{O}$, the space M_ζ is invariant under the operator $\mathrm{Ind}_\delta^G(j(f))$: if $\psi \in M_\zeta$, and $x \notin \zeta$,

$$\begin{aligned}
[\mathrm{Ind}_\delta^G(j(f))\psi](x) &= \sum_{y \in G^v} j(f)(xy)\psi(y^{-1})\sigma(xy, y^{-1}) \\
&= \sum_{y \in G_v} j(f)(xy^{-1})\psi(y)\sigma(xy^{-1}, y) \\
&= \sum_{y \in G_v \cap \zeta} j(f)(xy^{-1})\psi(y)\sigma(xy^{-1}, y) \\
&= 0
\end{aligned}$$

because if $y \in \zeta$ and $x \notin \zeta$, then $xy^{-1} \notin H$ hence $j(f)(xy^{-1}) = 0$.

Let $c : \mathcal{O} \to G_v$ be a map assigning to each orbit $\zeta \in \mathcal{O}$ an element $c(\zeta) \in \zeta$. Note that the orbit ζ_v of v itself is precisely H_v, since

$$\zeta_v = \{h \cdot v \mid d(h) = r(v)\} = \{hv \mid d(h) = v\} = \{h \mid d(h) = v\} = H_v.$$

We require that $c(H_v) = v$.

We now define operators

$$V_\zeta : \ell^2(H_{r(c(\zeta))}) \to \ell^2(G_v)$$

as follows. Fix ζ. Denote $x_0 = c(\zeta)$ and $u = r(x_0)$. Notice that if $y \in \zeta$ then $y = hx_0$ where $h \in H_u$. Thus $yx_0^{-1} = hx_0x_0^{-1} = hr(x_0) = hu = h \in H_u$. We define for $f \in \ell^2(H_u)$ and $y \in G_v$

$$[V_\zeta(f)](y) = \begin{cases} f(yx_0^{-1})\sigma(yx_0^{-1}, x_0) & y \in \zeta \\ 0 & \text{otherwise} \end{cases}$$

It is clear that $V_\zeta(f) \in M_\zeta$, and it follows that $\ell^2(G_v) = \bigoplus_{\zeta \in \mathcal{O}} Ran(V_\zeta)$.

We claim that

$$V_\zeta^* : \ell^2(G_v) \to \ell^2(H_u)$$

is given by

$$[V_\zeta^*(g)](h) = g(hx_0)\sigma(h, x_0)^{-1},$$

for $g \in \ell^2(G_v)$ and $h \in H_u$. Indeed,

$$\begin{aligned}
\langle V_\zeta(f), g\rangle_{\ell^2(G_v)} &= \int_{G_v} [V_\zeta(f)](y)\overline{g(y)}dy \\
&= \int_\zeta f(yx_0^{-1})\sigma(yx_0^{-1}, x_0)\overline{g(y)}dy
\end{aligned}$$

If we now write $y = hx_0$ and $h = yx_0^{-1} \in H_u$, we obtain:

$$\begin{aligned} &= \int_{H_u} f(h)\sigma(h, x_0)\overline{g(hx_0)}dh \\ &= \int_{H_u} f(h)\overline{V_\zeta^*(g)(h)}dh \\ &= \langle f, V_\zeta^*(g)\rangle_{\ell^2(H_u)} \end{aligned}$$

It is now easy to see that $V_\zeta^* V_\zeta = 1$ hence V_ζ is an isometry:

$$\begin{aligned} [(V_\zeta^* V_\zeta)(f)](h) &= [V_\zeta(f)](hx_0)\sigma(h, x_0)^{-1} \\ &= f(hx_0x_0^{-1})\sigma(hx_0x_0^{-1}, x_0)\sigma(h, x_0)^{-1} \\ &= f(h)\sigma(h, x_0)\sigma(h, x_0)^{-1} \\ &= f(h). \end{aligned}$$

We claim that the following identity holds:

$$\operatorname{Ind}_\delta^G(j(f)) = \sum_{\zeta\in\mathcal{O}} V_\zeta \operatorname{Ind}_{\delta_{r(c(\zeta))}}^H(f)V_\zeta^*. \tag{1}$$

Since the subspaces M_ζ are invariant under $\operatorname{Ind}_\delta^G(j(f))$, it suffices to show that given $\zeta \in \mathcal{O}$ and $\psi \in Ran(V_\zeta) = M_\zeta$,

$$\operatorname{Ind}_\delta^G(j(f))\psi = V_\zeta \operatorname{Ind}_{\delta_{r(c(\zeta))}}^H(f)V_\zeta^*\psi.$$

So fix ζ, and write $\psi = V_\zeta(g)$ where $g \in \ell^2(H_u)$. Take $x \in \zeta$:

$$\begin{aligned} [\operatorname{Ind}_\delta^G(j(f))\psi](x) &= [\operatorname{Ind}_\delta^G(j(f))V_\zeta(g)](x) \\ &= \sum_{y\in G^v} j(f)(xy)V_\zeta(g)(y^{-1})\sigma(xy, y^{-1}) \\ &= \sum_{y\in G^v, y^{-1}\in\zeta} j(f)(xy)g(y^{-1}x_0^{-1})\sigma(y^{-1}x_0^{-1}, x_0)\sigma(xy, y^{-1}) \\ &= \sum_{y\in G^v, y^{-1}\in\zeta} f(xy)g(y^{-1}x_0^{-1})\sigma(y^{-1}x_0^{-1}, x_0)\sigma(xy, y^{-1}) \end{aligned}$$

Since $y^{-1} \in \zeta$ write $y^{-1} = h^{-1}x_0$ for $h^{-1} \in H_u$, so $y = x_0^{-1}h$ and we obtain:

$$\begin{aligned} &= \sum_{h\in H^u} f(xx_0^{-1}h)g(h^{-1}x_0x_0^{-1})\sigma(h^{-1}x_0x_0^{-1}, x_0)\sigma(xx_0^{-1}h, h^{-1}x_0) \\ &= \sum_{h\in H^u} f(xx_0^{-1}h)g(h^{-1})\sigma(h^{-1}, x_0)\sigma(xx_0^{-1}h, h^{-1}x_0) \end{aligned}$$

Invoking the 2-cocycle property and then noticing that $hh^{-1} = r(h) = u = r(x_0)$, we obtain:

$$\begin{aligned}
&= \sum_{h\in H^u} f(xx_0^{-1}h)g(h^{-1})\sigma(xx_0^{-1}hh^{-1}, x_0)\sigma(xx_0^{-1}h, h^{-1}) \\
&= \sigma(xx_0^{-1}, x_0)\sum_{h\in H^u} f(xx_0^{-1}h)g(h^{-1})\sigma(xx_0^{-1}h, h^{-1}) \\
&= \sigma(xx_0^{-1}, x_0)[\operatorname{Ind}_{\delta_u}^H(f)g](xx_0^{-1}) \\
&= [V_\zeta \operatorname{Ind}_{\delta_u}^H(f)g](x) \\
&= [V_\zeta \operatorname{Ind}_{\delta_u}^H(f)V_\zeta^*\psi](x)
\end{aligned}$$

If $x \notin \zeta$ then both sides of the equation are zero. Thus we have proven (1).

Now note that $\|\operatorname{Ind}_{\delta_v}^G(j(f))\| = \|\sum_{\zeta\in\mathcal{O}} V_\zeta \operatorname{Ind}_{\delta_{r(c(\zeta))}}^H(f)V_\zeta^*\| = \sup_\zeta\{\|\operatorname{Ind}_{\delta_{r(c(\zeta))}}^H(f)\|\}$. Moreover, recall that we required that $c(H_v) = v$. Therefore, when δ_v ranges over all unit point masses on $G^{(0)}$, so does $\delta_{r(c(\zeta))}$. Recalling that $H^{(0)} = G^{(0)}$, we finally conclude that:

$$\begin{aligned}
\|j(f)\|_{C_r^*(G,\sigma)} &= \sup\{\|\operatorname{Ind}_{\delta_v}^G(j(f))\| : \delta_v \text{ is a unit point mass on } G^{(0)}\} \\
&= \sup\{\|\operatorname{Ind}_{\delta_v}^H(f)\| : \delta_v \text{ is a unit point mass on } H^{(0)}\} \\
&= \|f\|_{C_r^*(H,\sigma)}
\end{aligned}$$

We have thus shown that j is an isometric $*$-homomorphism with respect to the reduced C^*-norms. It follows from the general theory that j extends to an isometric $*$-homomorphism $j : C_r^*(H,\sigma) \to C_r^*(G,\sigma)$.

□

Proposition 1 generalizes the following result of S. Kaliszewski, J. Quigg and I. Raeburn.

Proposition 2. *(*[3]*, Lemma 5.8) Let c be a continuous homomorphism of an r-discrete Hausdorff groupoid Q into a discrete group G, and put $N = c^{-1}(e)$. Assume that Q is amenable. Then the canonical map $i : C^*(N) \to C^*(Q)$ is faithful.*

Proof. Since G is discrete, $\{e\}$ is open. Therefore $N = c^{-1}(e)$ is open in Q, since c is continuous. By proposition 5.1.1 in [1], we know that N is amenable, being an open subgroupoid of the amenable groupoid Q. The assertion now follows from Proposition 1 above, and from the fact that for amenable groupoids, the reduced and the full C^*-algebras coincide. □

We remark that similar maps were considered also in [2], §6.

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References

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