MATH 009B (053)
Quiz 2 Solutions

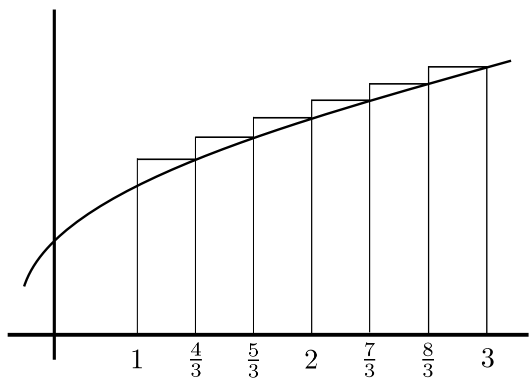
Problem 1

Approximate the integral

$$\int_1^3 \frac{x^2}{x^2+1}\,dx$$

using a right-hand Riemann sum with 6 endpoints. Leave your answer as an unsimplified sum of 6 terms.

Solution: According to the directions, we need to have 6 rectangles to compute 6 areas, hence $n = 6$. Let the endpoints be $a = 1$ and $b = 3$. We must recall the definition of the width of each rectangle, which is given as $\Delta x = \frac{b-a}{n} = \frac{3-1}{6} = \frac{2}{6} = \frac{1}{3}$. It isn't necessary to draw the picture to do this problem, but it helps in understanding the problem. The graph is given below. Since we



are doing a RH Riemann sum, we start at the right endpoint, draw up to the function, then draw a line over to the next point and draw a line down, and repeat. The figure show where we evaluate the function, all the points where the function intersects the corner of the rectangle. Here is the math:

$$\begin{aligned}\sum_{i=1}^{6} f(x_i)\Delta x &= \Delta x \sum_{i=1}^{6} f(x_i)\\ &= \Delta x \left(\frac{\left(\frac{4}{3}\right)^2}{\left(\frac{4}{3}\right)^2+1} + \frac{\left(\frac{5}{3}\right)^2}{\left(\frac{5}{3}\right)^2+1} + \frac{(2)^2}{(2)^2+1} + \frac{\left(\frac{7}{3}\right)^2}{\left(\frac{7}{3}\right)^2+1} + \frac{\left(\frac{8}{3}\right)^2}{\left(\frac{8}{3}\right)^2+1} + \frac{(3)^2}{(3)^2+1} \right)\\ &= \frac{1}{3}\left(\frac{16}{25} + \frac{25}{34} + \frac{4}{5} + \frac{49}{58} + \frac{64}{73} + \frac{9}{10} \right)\end{aligned}$$

Problem 2

Use the Fundamental Theorem of Calculus to evaluate: $\dfrac{d}{dx}\left(\int_0^{\tan(x)} e^{-t^2}\ dt\right)$

Solution: The key to these problems is to remember the chain rule on the bounds of integration. First I will show the detailed explanation which would for sure get you full credit on any exam. The second method is the quick shortcut without all the details.

Method 1 (Detailed-Rigorous Version): Essentially this question is asking you to find $F'(x)$, where

$$F(x) = \int_0^{\tan(x)} e^{-t^2}\ dt$$

First, we want to view $F(x)$ in a slightly different way, the way the Fundamental Theorem of Calculus can be applied the way we know it. Thus view $F(x)$ as

$$G(x) = \int_0^{x} e^{-t^2}\ dt,$$

then we know from the Fundamental Theorem of Calculus that $G'(x) = e^{-x^2}$. Now, our question didn't have just an x, it had $\tan(x)$. Thus, $F(x)$ is a composition of $G(x)$ and $g(x) = \tan(x)$, which means that $F(x) = G(g(x))$. Also, we will need the fact that $g'(x) = (\tan(x))' = \sec^2(x)$. Then by our usual 009A calculus chain rule, we have the following

$$\begin{aligned} F'(x) &= G'(g(x))g'(x) \\ &= e^{-(g(x))^2} g'(x) \\ &= e^{-\tan^2(x)} \sec^2(x) \end{aligned}$$

Method 2 (Quick Version): For the shortcut method, we can write the general situation as

$$F(x) = \int_{h(x)}^{g(x)} f(t)\ dt$$

The formula for the derivative is given below.

$$F'(x) = \frac{d}{dx}\left(\int_{h(x)}^{g(x)} f(t)\ dt\right) = f(g(x))g'(x) - f(h(x))h'(x)$$

Some explanation is in order. To remember this, just compute by taking the composition of the top bound $g(x)$ and the integrand $f(t)$, and multiplying by the derivative of the top bound, $g'(x)$. Then subtract by the composition of the bottom bound $h(x)$ and the integrand $f(t)$, and multiply by the derivative of the bottom bound, $h'(x)$. Having this formula, we just apply directly to our problem

$$\begin{aligned} F'(x) = \frac{d}{dx}\left(\int_0^{\tan(x)} e^{-t^2}\ dt\right) &= e^{-\tan^2(x)} \sec^2(x) - e^{-\tan^2(x)}(0) \\ &= e^{-\tan^2(x)} \sec^2(x) \end{aligned}$$

NOTE: You can only apply this method when the domains of the functions $g(x)$ and $h(x)$, overlap. Otherwise, you cannot apply this method.