Review: Limits, Indeterminate Forms & Improper Integrals

The point of the review is to go over important concepts from 009A and 009B that will be very helpful for this course.

Recall the 2 examples usually shown:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \frac{0}{0} \quad \text{and} \quad \lim_{x\to\infty} \frac{x+1}{x-1} = \frac{\infty}{\infty}$$

These expressions are indeterminate forms, $\frac{0}{0}$ and $\frac{\infty}{\infty}$.
Indeterminate forms in themselves tell us nothing!
The limit could be 0, ∞, or any finite number.
More work is required to find the value of the limit.

Indeterminate Forms: $0/0$, $\frac{\infty}{\infty}$, $\infty\cdot 0$, $\infty-\infty$, 0^0, 1^∞, ∞^0

Classic 009A way to deal with indeterminate forms

L'Hôpital's Rule: f, g differentiable, $g'(x) \neq 0$. Suppose f,g diff on (α,β) except maybe $a \in (\alpha,\beta)$

① $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then

② $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if limit on RHS is $\pm\infty$ or limit exists.

Note: We can only use L'Hopital's Rule for $\frac{0}{0}$ or $\frac{\infty}{\infty}$ cases!! The other indeterminate forms must be transformed to one of these to use L'H.

Examples: $\left(\frac{d}{dx}\sec^2 x = 2\sec^2(x)\tan(x)\right)$

① $\lim\limits_{x\to 1} \dfrac{\ln(x)}{x-1}$

② $\lim\limits_{x\to\infty} \dfrac{e^x}{x^2}$

③ $\lim\limits_{x\to\infty} \dfrac{\tan(x)-x}{x^3}$

④ $\lim\limits_{x\to 0^+} x\ln x$

⑤ $\lim\limits_{x\to(\frac{\pi}{2})^-} \sec(x)-\tan(x)$

⑥ $\lim\limits_{x\to 0^+} \left(1+\sin(4x)\right)^{\cot(x)}$

⑦ $\lim\limits_{x\to 0^+} x^x$

⑧ $\lim\limits_{x\to\infty} x^3 e^{-x^2}$

Solution ⑥ $\lim\limits_{x\to 0^+} \left(1+\sin(4x)\right)^{\cot(x)} = 1^\infty$ indeterminate

$$= \lim_{x\to 0^+} e^{\ln\left[\left(1+\sin(4x)\right)^{\cot(x)}\right]}$$

$$= \lim_{x\to 0^+} e^{\cot(x)\ln\left[\left(1+\sin(4x)\right)\right]} = e^{\lim\limits_{x\to 0^+} \frac{\ln(1+\sin(4x))}{\tan(x)}}$$

$$= e^{\lim\limits_{x\to 0^+} \frac{\frac{4\cos(4x)}{1+\sin(4x)}}{\sec^2(x)}} = e^{\lim\limits_{x\to 0^+} \frac{4\cos(4x)}{(1+\sin(4x))}\cdot\frac{\cos^2(x)}{1}}$$

$$= e^4$$

Solution ⑦ $\lim\limits_{x\to 0^+} x^x = 0^0$ indeterminate

Let $y = x^x \Rightarrow \ln(y) = \ln(x^x) = x\ln(x)$

$$\Rightarrow \lim_{x\to 0^+} \ln(y) = \lim_{x\to 0^+} x\ln(x)$$

$$\Rightarrow e^{\lim\limits_{x\to 0^+} \ln(y)} = e^{\lim\limits_{x\to 0^+} x\ln x}$$

$$\lim_{x\to 0^+} y = e^0 = \boxed{1}$$

Review - Improper Integrals

There are two types of integrals (improper) that we consider in 009B, ones with infinite ($\pm\infty$) bounds and ones with finite bounds and discontinuities in the integrand. We will only consider the former.

Recall we always rewrite these in terms of limits:

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx \quad \text{provided the limit exists}$$

Then if possible, we compute the integral, then take a limit.

Examples: ① $\int_1^\infty \frac{1}{x}\,dx$ ③ $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$

② $\int_{-\infty}^0 xe^x\,dx$ ④ $\int_1^\infty \frac{1}{x^p}\,dx$ (for which p?!) (conv. $p>1$, div $p\le 1$)

Comparison Theorem; What if we cannot do the integration directly? We need another method.

Suppose f, g are continuous s.t. $f(x) \geq g(x)$ (≥ 0) $\forall x \geq a$

① If $\int_a^\infty f(x)\,dx$ converges $\Rightarrow \int_a^\infty g(x)\,dx$ is convergent

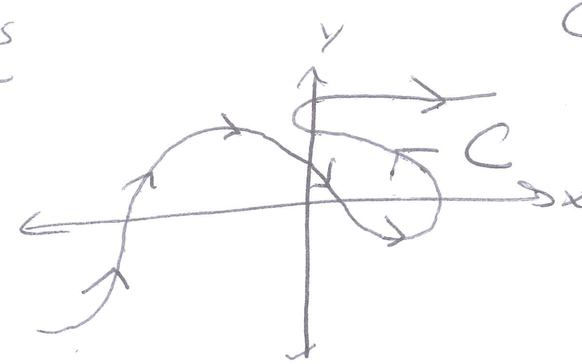
② If $\int_a^\infty g(x)\,dx$ diverges $\Rightarrow \int_a^\infty f(x)\,dx$ is divergent

Ex) ① $\int_1^\infty e^{-x^2}\,dx$ convergent or divergent? Compare with e^{-x} (can we integrate? See 11.4)

② $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ (show $\frac{1}{x} < \frac{1+e^{-x}}{x}$)

③ $\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\,dx$ ($\sec(x) > 1$ for $0<x<1$ $\Rightarrow \sec^2(x) > 1$) compare with $\frac{1}{x^{3/2}}$

Section 9.2 - Parametric Equations



Imagine a particle moving along along a curve C shown to the right. We cannot describe the curve C by a function $y=f(x)$, as it fails the "vertical line test."

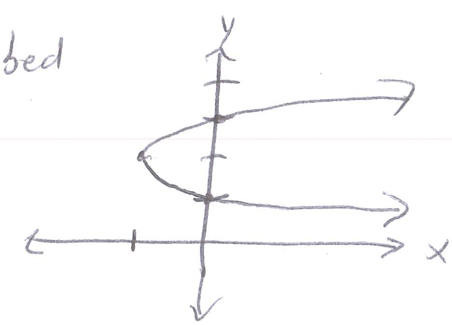
Instead, we introduce a third variable, say "t", which is called the parameter. So for a given t, we can construct two functions

$$x = f(t) \quad \text{and} \quad y = g(t)$$

which produce the x and y coordinates of a graph. These equations are parametric equations. Each ~~(x,y)~~ (x,y) in $\mathbb{R}^2$ which can trace out C. We then call C a parametric curve

Ex) Sketch and identify the curve described by
$x = t^2 - 2t$
$y = t+1$

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3



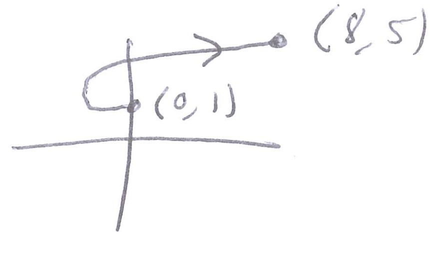
a sideways parabola

We can "eliminate the parameter" t

$x = t^2 - 2t$
$= (y-1)^2 - 2(y-1)$
$x = y^2 - 4y + 3 \Rightarrow$ parabola

We can also restrict values of t to get a portion of the graph.

$$\begin{cases} x = t^2 - 2t \\ y = t + 1 \end{cases} \quad 0 \leq t \leq 4 \quad \Rightarrow$$



So if we have $a \leq t \leq b$

$(f(a), g(a))$ is the initial point

$(f(b), g(b))$ is the terminal point

So the graph has a direction

Eliminate the parameter

Ex) $\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases} \Rightarrow \begin{matrix} x^2 = \cos^2(t) \\ y^2 = \sin^2(t) \end{matrix} \Rightarrow x^2 + y^2 = \cos(t) + \sin(t)$

$\boxed{x^2 + y^2 = 1}$ circle $r = 1$ center $(0,0)$

Ex) $\begin{cases} x = h + r\cos(t) \\ y = k - r\sin(t) \end{cases} \Rightarrow \begin{matrix} x - h = r\cos(t) \\ y - k = r\sin(t) \end{matrix} \Rightarrow \begin{matrix} (x-h)^2 = r^2\cos^2(t) \\ (y-k)^2 = r^2\sin^2(t) \end{matrix}$

$\Rightarrow (x-h)^2 + (y-k)^2 = r^2(\cos^2(t) + \sin^2(t))$

$\Rightarrow \boxed{(x-h)^2 + (y-k)^2 = r^2}$

Ex) $\begin{cases} x = \dfrac{1}{t^2+1} \\ y = \dfrac{t^2}{t^2+1} \end{cases} \Rightarrow xt^2 + x - 1 = 0$ or $\dfrac{1}{x} - 1 = t^2$ so $t = \pm\sqrt{\dfrac{1}{x} - 1}$

$\Rightarrow y = \dfrac{1/x - 1}{1/x - 1 + 1} = \dfrac{\frac{1}{x} - 1}{1/x} = \boxed{1 - x}$

So we have a line $y = 1 - x$.

Convert rectangular Cartesian to parametric

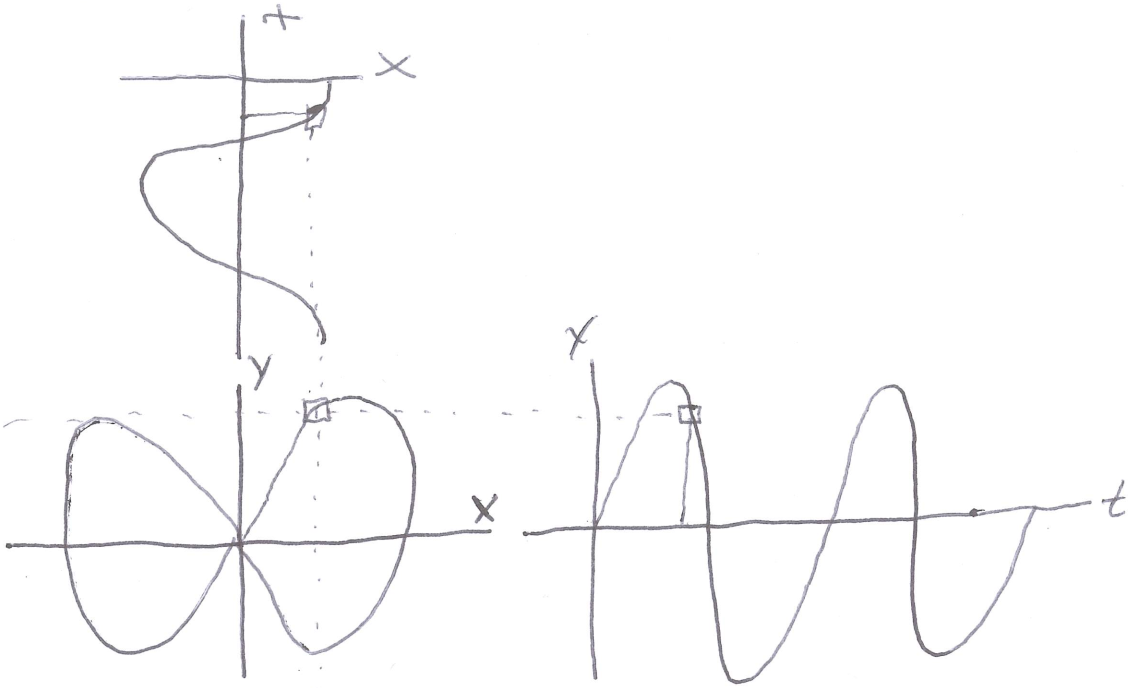
Let $y = x^2$, find f, g s.t. $x = f(t)$ and $y = g(t)$, where $\dfrac{dy}{dx} = t$

So $\dfrac{dy}{dx} = 2x \Rightarrow 2x = t \Rightarrow x = \dfrac{t}{2} \Rightarrow y = x^2 = \dfrac{t^2}{4}$

So $\boxed{\begin{matrix} x = \dfrac{t}{2} \\ y = \dfrac{t^2}{4} \end{matrix}}$

Sketching graphs

Sketch $x = \cos(t)$, $y = \sin(2t)$



Ex) $x = \sqrt{t}$ $\quad t \geq 0$ $\left(\Rightarrow y = t = (\sqrt{t})^2 = x^2 \quad \begin{array}{l} t \geq 0 \\ \Rightarrow x \geq 0 \end{array} \right)$

$y = t$

Ex) $x = t$ $\quad$ all t

$y = t^2$

Ex) $x = t + \frac{1}{t}$ $\quad t > 0$ $\left(\begin{array}{c} \text{Find } x-y \text{ and } x+y \\ (x-y)(x+y) \Rightarrow x^2 - y^2 = 4 \end{array} \right)$

$y = t - \frac{1}{t}$

Section 9.3 - Calculus on Parametric Curves

Given $x=f(t), y=g(t)$ how do we find $\frac{dy}{dx}$?

$$\Rightarrow \boxed{\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{or} \quad \frac{dy}{dx}=\frac{g'(t)}{f'(t)}}$$

if $\frac{dx}{dt}\neq 0$

How? Chain Rule!

$$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}$$

So, we can see there is a

1. Horizontal Tangent Line when $\frac{dy}{dt}=0$
2. Vertical Tangent Line when $\frac{dx}{dt}=0$ (and $\frac{dy}{dt}\neq 0$)

Also
$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{dy}{dx}\right]=\frac{d}{dx}\left[h(t)\right]=\frac{\frac{dh}{dt}}{\frac{dx}{dt}}$$

if we let $h(t)=\frac{dy}{dx}$

By chain rule again

$$\frac{dh}{dt}=\frac{dh}{dx}\cdot\frac{dx}{dt}$$
$$\Rightarrow \frac{dh}{dx}=\frac{\frac{dh}{dt}}{\frac{dx}{dt}}$$

$$\frac{d^2y}{dx^2}=\boxed{\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}}$$

Ex) Let C be defined as: $x=t^2$
$y=t^3-3t$

a) show C has two tangents at $(3,0)$.

b) Find horiz. and vert. tangent lines.

c) Determine concavity.

d) sketch.

Solution

a) $y = t^3 - 3t = t(t^2-3) = 0 \qquad x = t^2 \Rightarrow x = 3$ twice

$t=0,\ t = \pm\sqrt{3}$

$$\frac{dy}{dx} = \frac{3t^2-3}{2t} = \frac{3}{2}\left(t - \frac{1}{t}\right) = \pm\frac{6}{2\sqrt{3}} = \pm\sqrt{3}$$

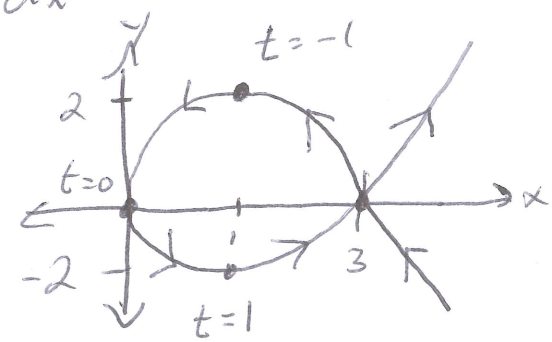
$$\Rightarrow \boxed{\begin{aligned} y &= \sqrt{3}(x-3) \\ y &= -\sqrt{3}(x-3) \end{aligned}}$$

b) $\dfrac{dy}{dx} = \dfrac{3t^2-3}{2t} = \dfrac{3(t^2-1)}{2t} \Rightarrow$ Horiz: $3(t^2-1)=0$, $t = \pm 1 \Rightarrow (1,-2)$, $(1,2)$

Vert: $2t = 0$, $t = 0 \Rightarrow (0,0)$

c) $\dfrac{d^2y}{dx^2} = \dfrac{3(t^2+1)}{4t^3} \Rightarrow$ C.C. up $t>0$; C.C. down $t<0$

d)



Ex) Find tangent line to $\begin{cases} x = r(t - \sin(t)) \\ y = r(1-\cos(t)) \end{cases}$ at $t = \dfrac{\pi}{3}$, r constant

and Horizontal/Vertical tangent line points

Solution: $\dfrac{dy}{dx} = \dfrac{r\sin(t)}{r(1-\cos(t))} = \dfrac{\sin(t)}{1-\cos(t)} \Rightarrow \dfrac{dy}{dx}(\pi/3) = \dfrac{\sin(\pi/3)}{1-\cos(\pi/3)} = \sqrt{3}$

$x = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$

$y = r(1-\cos(\pi/3)) = \frac{r}{2}$

$$\Rightarrow \boxed{y - \frac{r}{2} = \sqrt{3}\left(x - \left(r\left(\pi/3 - \frac{\sqrt{3}}{2}\right)\right)\right)}$$

(b) Horiz: $\sin(t) = 0$ and $\cos(t) \neq 1 \Rightarrow (2n-1)\pi$ $n \in \mathbb{Z}$

Coordinates: $((2n-1)\pi, 2r)$

Vert: $\cos(t) = 1 \Rightarrow 2n\pi$ for $n \in \mathbb{Z}$

But what about when $t = 2n\pi$ for $\sin(t)$ and $\cos(t)$? we have $\frac{0}{0}$!!

$$\lim_{t \to 2n\pi^+} \frac{dy}{dx} = \lim_{t \to 2n\pi^+} \frac{\sin(t)}{1-\cos(t)} = \lim_{t \to 2n\pi^+} \frac{\cos(t)}{\sin(t)} = +\infty$$

$\sin t > 0$ for small $t > 0$

$$\lim_{t \to 2n\pi^-} \frac{dy}{dx} = \lim_{t \to 2n\pi^-} \frac{\sin(t)}{1-\cos(t)} = \lim_{t \to 2n\pi^-} \frac{\cos(t)}{\sin(t)} = -\infty$$

$\sin t < 0$ for small $t < 0$

Arc Length

Recall from 009B, that Arc Length is given by

by using differentials $dt = \frac{1}{f'(t)} dx$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \int_a^b \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2} \frac{dx}{dt} \cdot dt$$

$$= \int_a^b \sqrt{\frac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}} \frac{dx}{dt} \cdot dt$$

$$\boxed{L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt}$$

Ex) Find Arc length of $\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases}$ (we know its a circle! by before)

Ex) For $\begin{cases} x = \cos(2t) \\ y = \sin(2t) \end{cases}$

Ex) For $\begin{cases} x = r(t - \sin(t)) \\ y = r(1 - \cos(t)) \end{cases}$ (Ans: $8r$)

r const.

Identity: $2(1-\cos(t)) = 4\sin^2\left(\frac{t}{2}\right)$

Surface Area

For a solid of revolution, we can find its surface area, like in 009B.

If $x = f(t)$ and $y = g(t)$ then

① Surface created by rotating the graph about the x-axis (where $g(t) \geq 0$ on $[t_1, t_2]$):

$$SA = 2\pi \int_{t_1}^{t_2} g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

② Surface created by rotating the graph about the y-axis (where $f(t) \geq 0$ on $[t_1, t_2]$):

$$SA = 2\pi \int_{t_1}^{t_2} f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Ex) Rotate $\begin{cases} x = r\cos(t) \\ y = r\sin(t) \end{cases}$ $0 \leq t \leq \pi$ about the x-axis

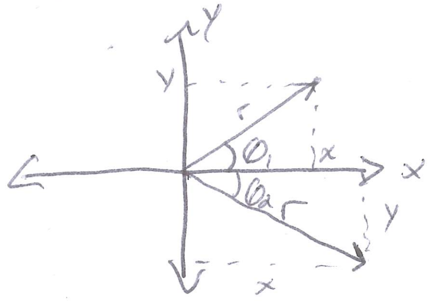
(Ans. $4\pi r^2$)

Ex) Rotate $\begin{cases} x = \cos^3(t) \\ y = \sin^3(t) \end{cases}$ about the x-axis $0 \leq t \leq \pi/2$ (Ans. $\frac{6\pi}{5}$)

(use substitution $u = \sin(t)$)

Section 9.4 - Polar Coordinates

Instead of using (x,y) coordinates in $\mathbb{R}^2$, we can use (r,θ) to describe the same point in $\mathbb{R}^2$.



Given a point (x,y) we can find the polar representation via

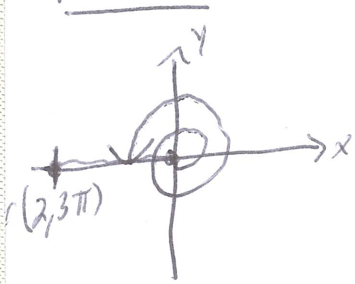
$$\boxed{r^2 = x^2+y^2 \quad \text{and} \quad \tan\theta = y/x}$$

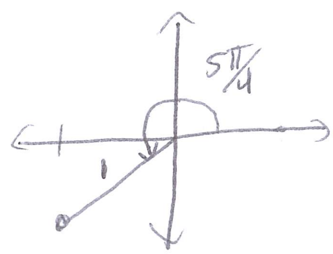
Given a point (r,θ) we can find ~~that~~ it's Cartesian representation via

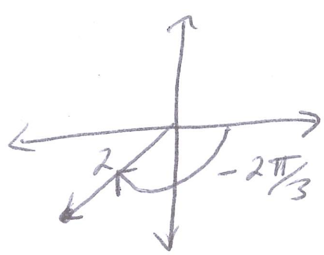
$$x = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta)$$

Note: $r \geq 0$

Plot: $(2,3\pi)$, $(1, 5\pi/4)$, $(2, -2\pi/3)$







Convert: $(2, \pi/3)$ to Cartesian: $(1, \sqrt{3})$

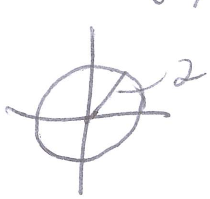
$(+1,-1)$ to polar : $(\sqrt{2}, -\pi/4)$

$(\sqrt{2}, 7\pi/4)$

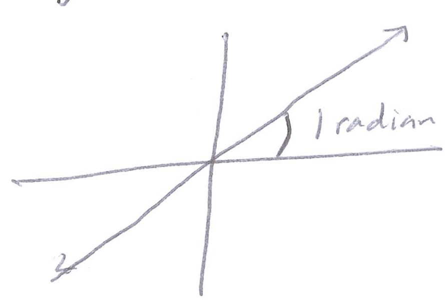
Graphs are now given as $r = f(\theta)$

Ex) $r=2 \Rightarrow$ all points s.t. $(2,\theta)\ \forall \theta$

circle



Ex) $\theta = 1$ $(r,1)$



Ex) $r = 2\cos(\theta)$

Plot points, or

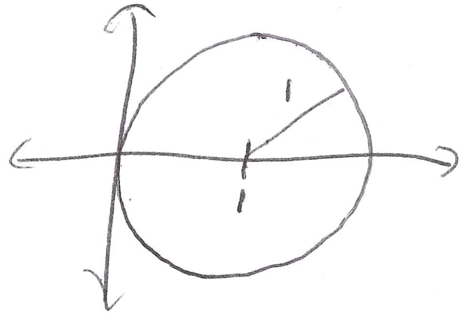
$\cos(\theta) = \frac{r}{2}$

$\frac{x}{r} = \frac{r}{2}$

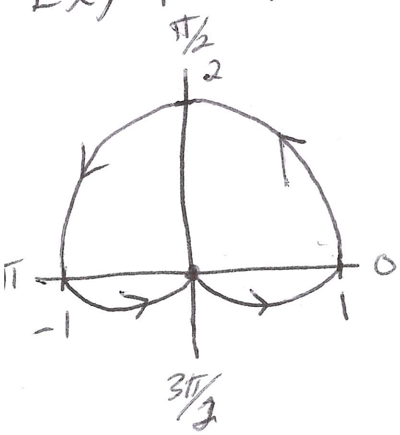
$r^2 = 2x$

$x^2 + y^2 = 2x$

$(x-1)^2 + y^2 = 1$

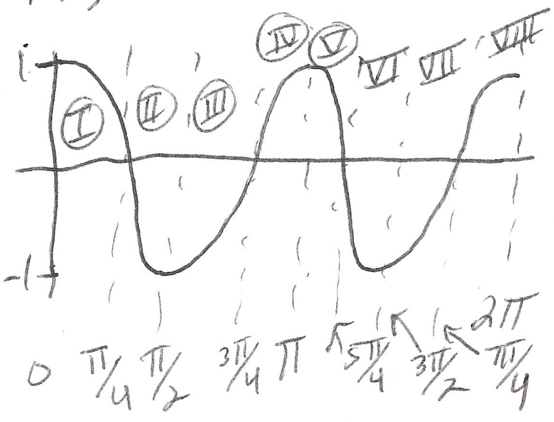


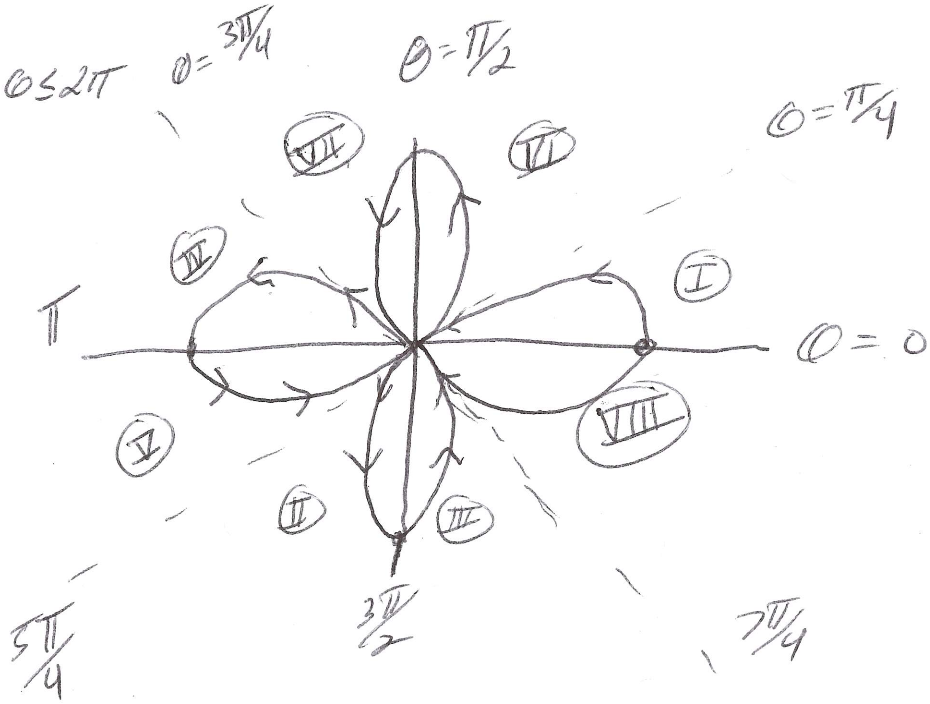
Ex) $r = 1 + \sin(\theta)$



Cardoid

Ex) $r = \cos(2\theta)$ $\quad 0 \le \theta \le 2\pi$





Section 9.5 - Calculus with Polar Graphs

Tangent Lines

Recall polar coordinate formulas: $x = r\cos(\theta)$ $\quad y = r\sin(\theta)$

$x = f(\theta)\cos(\theta)$ $\quad y = f(\theta)\sin(\theta)$

and $r = f(\theta)$

$$\Longrightarrow \begin{cases} \dfrac{dx}{d\theta} = f'(\theta)\cos(\theta) \cancel{+} - f(\theta)\sin(\theta) \\ \dfrac{dy}{d\theta} = f'(\theta)\sin(\theta) + f(\theta)\cos(\theta) \end{cases}$$

or

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)$$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)$$

Then by chain rule (like parametric $\frac{dy}{dx}$)

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)}}$$

Ex) Let $r = 1 + 2\sin(\theta)$ $\quad 0 \le \theta \le 2\pi$

① Find tangent line at $\pi/4 = \theta$

② Find θ where there is horiz or vert. tangent lines

Solution: ① $\dfrac{dy}{dx} = \dfrac{2\cos(\theta)\sin(\theta) + \cos(\theta)(1+2\sin(\theta))}{2\cos^2(\theta) - \sin(\theta)(1+2\sin(\theta))}$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = -2\sqrt{2} - 1$$

$y_0 = r_0\sin(\theta_0)$

$y_0 = (1+\sqrt{2})\frac{\sqrt{2}}{2} = 1 + \frac{\sqrt{2}}{2}$

$x_0 = r_0\cos(\theta_0)$

$x_0 = (1+\sqrt{2})\frac{\sqrt{2}}{2} = 1 + \frac{\sqrt{2}}{2}$

$y - y_0 = (x - x_0)m$ $\Rightarrow$ $\boxed{y = (-2\sqrt{2}-1)\left(x - \left(1 + \frac{\sqrt{2}}{2}\right)\right) - \left(1 + \frac{\sqrt{2}}{2}\right)}$

$m = -2\sqrt{2} - 1$

(b) Horiz: $\frac{dy}{dx}=0 \Rightarrow \frac{dy}{d\theta}=0 \Rightarrow \cos(\theta)(4\sin(\theta)+1)=0$

$\boxed{\theta = \pi/2,\ 3\pi/2}$ since $\cos(\theta)=1$

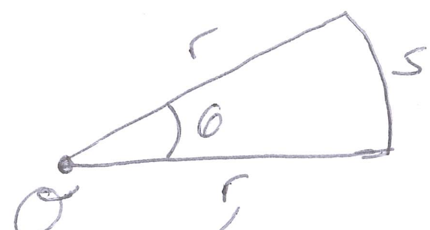
$\boxed{\theta = \sin^{-1}(-1/4)}$ since $\sin(\theta) = -1/4$

Vert: $\frac{dx}{d\theta} = 0 \Rightarrow 2(\cos^2(\theta) - \sin^2(\theta)) - \sin(\theta) = 0$

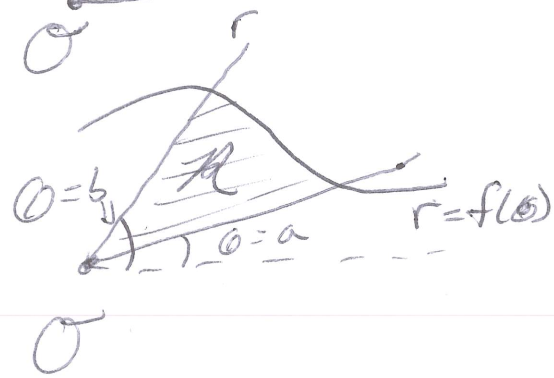
$\cos^2\theta = 1 - \sin^2\theta$

$$2(1-2\sin^2(\theta)) - \sin(\theta) = 0$$
$$-4\sin^2(\theta) - \sin(\theta) + 2 = 0$$
$$4\sin^2(\theta) + \sin(\theta) - 2 = 0$$
$$\sin(\theta) = \frac{-1}{2(4)} \pm \frac{\sqrt{1-4(4)(-2)}}{2(4)}$$
$$\sin(\theta) = -\frac{1}{8} \pm \frac{\sqrt{33}}{8}$$
$$\boxed{\theta = \sin^{-1}\left(\frac{-1 \pm \sqrt{33}}{8}\right)}$$

Areas and Lengths



Recall the area of a sector: $A = \frac{1}{2}r^2\theta$



For a general area, say the region labeled as $\mathcal{R}$ in the figure, then

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2 \, d\theta \quad \text{or} \quad A = \int_a^b \frac{1}{2} r^2 \, d\theta$$

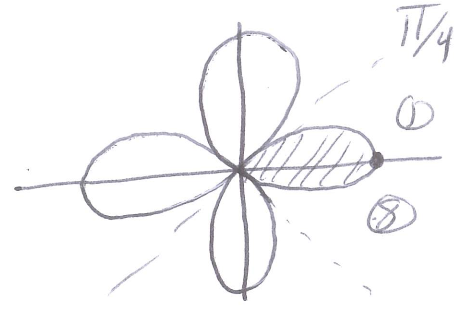
with the understandings that

$r = f(\theta)$!!

Find the area of one loop of

$r = \cos(2\theta)$

Recall the picture from last time



③

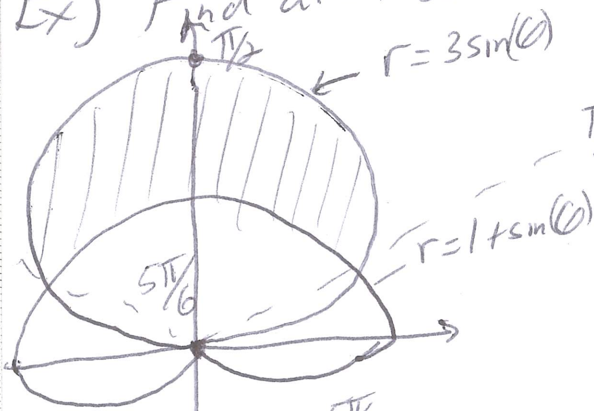
From the fact that for $\theta \in [\frac{-\pi}{4}, \frac{\pi}{4}]$ we get one petal, then

$$A = \frac{1}{2}\int_{-\pi/4}^{\pi/4} r^2\, d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2(2\theta)\, d\theta = 2\left(\frac{1}{2}\int_0^{\pi/4} \cos^2(2\theta)\, d\theta\right)$$

$$= \int_0^{\pi/4} \frac{1}{2}(1+\cos(4\theta))\, d\theta = \frac{1}{2}\left[\theta + \frac{1}{4}\sin(4\theta)\right]\Big|_0^{\pi/4}$$

$$= \boxed{\pi/8}$$

Ex) Find area between $r = 3\sin(\theta)$ (outside cardoid)
$r = 1+\sin(\theta)$



Solution: Find intersections first

$3\sin(\theta) = 1+\sin(\theta)$

$2\sin(\theta) = 1 \Rightarrow \sin(\theta) = \frac{1}{2}$

$\theta = \pi/6,\ 5\pi/6$

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin(\theta))^2\, d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1+\sin(\theta))^2\, d\theta$$

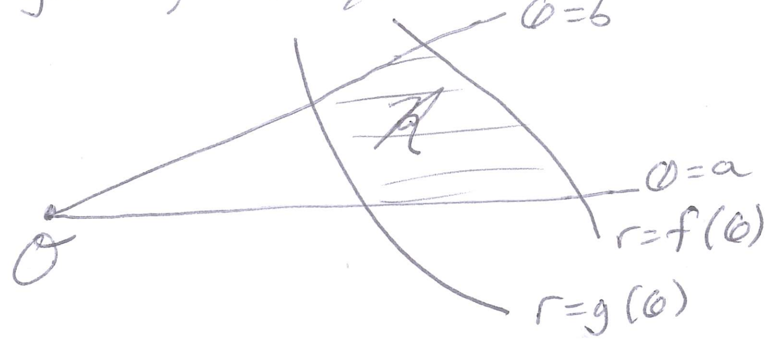
$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$

Symmetry

$$= 2\cdot\frac{1}{2}\left[\int_{\pi/6}^{\pi/2} (8\sin^2\theta - 2\sin\theta - 1)\, d\theta\right.$$

$$\boxed{A = \pi}$$

In general, if you have the followings



$$A = \frac{1}{2}\int_a^b \left((f(\theta))^2 - (g(\theta))^2 \right) d\theta$$

Example: Find the intersection points of $r = \cos(2\theta)$, $r = 1/2$

$$\Rightarrow \cos(2\theta) = \frac{1}{2}$$
$$\cos(u) = 1/2 \Rightarrow u = 2\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$$
$$\theta = \pi/6, 5\pi/6, \frac{7\pi}{6}, 11\pi/6$$

But there are 4 more!!

$$\cos(2\theta) = -\frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

Example: Practice Midterm #5

Arc Length: $r = f(\theta)$, $a \le \theta \le b$, $L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

Find arc length of $r = 1 + \sin(\theta)$ $0 \le \theta \le 2\pi$

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta$$
$$= \int_0^{2\pi} \sqrt{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta}\, d\theta$$
$$= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}$$
$$= 8$$

Use $x = \frac{\pi}{2} - y$

$$\sqrt{2}\int_{-3\pi/2}^{\pi/2} \sqrt{1+\cos(y)}\, dy$$
$$= 2\int_{-3\pi/2}^{\pi/2} |\cos(y/2)|\, dy$$
$$= 2\left[-\int_{-3\pi/2}^{-\pi} \cos(y/2)\, dy + \int_{-\pi}^{\pi/2} \cos(y/2)\, dy \right]$$

Section 8.1 - Sequences

A *sequence* is a list of numbers with a definite order

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

also denoted as $\{a_1, a_2, a_3, \ldots\}$, $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

Ex) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$, $\left\{\frac{(-1)^n(n+1)}{3^n}\right\}$, $\left\{\cos\left(\frac{n\pi}{2}\right)\right\}_{n=0}^{\infty}$

Ex) Find a formula, a_n, for the sequence $\left\{\frac{3}{5}, \frac{-4}{25}, \frac{5}{125}, \frac{-6}{625}, \ldots\right\}$

Ex) Recursive sequences: $f_1 = f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$ $n \geq 3$ Fibonacci sequence

Defn: A *sequence* has a limit, L, written as

(i) $\lim_{n\to\infty} a_n = L$ or (ii) $a_n \to L$ as $n \to \infty$

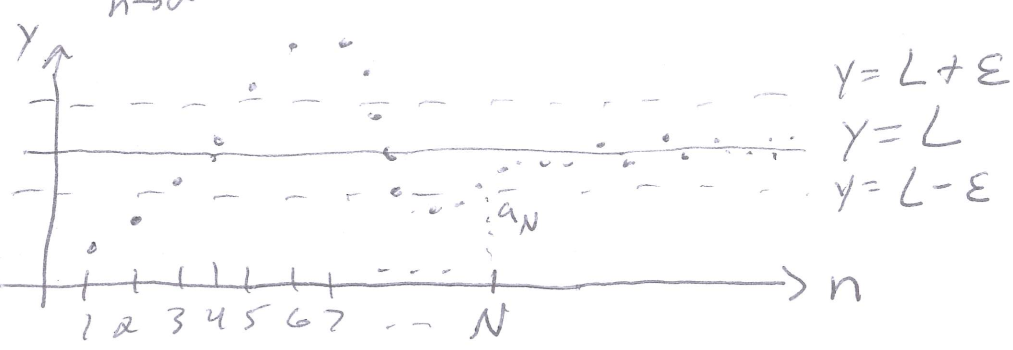
if a_n is as close to L as we like for sufficiently large n.

Rigorous: A sequence has a limit L if for every $\varepsilon > 0$ there exists an integer N s.t. for $n \geq N$,

$$|a_n - L| < \varepsilon.$$

Note: If $\lim_{n\to\infty} a_n$ exists, and is finite, we say the sequence *converges*.

If $\lim_{n\to\infty} a_n = \infty$ or DNE, then the sequence *diverges*.



Theorem: If $\lim_{x\to\infty} f(x) = L$, and $f(n) = a_n$, with $n \in \mathbb{Z}$, then

$$\lim_{n\to\infty} a_n = L$$

Defn: If $\lim_{n\to\infty} a_n = \infty \Rightarrow$ for every $M > 0$, there exists an integer N s.t. for $n \geq N$, $a_n \geq M$

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences we have:
sum rule, difference rule, product rule, quotient rule, power rule
i.e.) sums of sequences converge, products converge, etc.

Squeeze Theorem: If $a_n \leq b_n \leq c_n$ for $n \geq N$ and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L \implies \lim_{n\to\infty} b_n = L$$

Theorem: If $\lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$

Find Limits: $a_n = \frac{n}{n+1}$, $a_n = \frac{\ln(n)}{n}$, $a_n = (-1)^n$, $a_n = \frac{(-1)^n}{n}$

Theorem: If $\lim_{n\to\infty} a_n = L$ and f is a continuous function at L

then $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L)$

Ex) Find Limit $a_n = \frac{n!}{n^n}$ (Squeeze Thm $\frac{1}{n}\left(\frac{2\cdot 3\cdot 4\cdots n}{n\cdot n\cdot n\cdots n}\right)$)

$a_n = \sin(\pi/n)$

Geometric sequence: The sequence $\{r^n\}_{n=1}^{\infty}$, r = constant is a geometric sequence.

It is convergent for $|r| < 1$ and $r = 1$

or $-1 < r \leq 1$

A sequence is <u>increasing</u> if $a_n < a_{n+1}$ for all $n \geq 1$

A sequence is <u>decreasing</u> if $a_{n+1} < a_n$ for all $n \geq 1$

A sequence is <u>monotonic</u> if it is either increasing or decreasing.

Ex) Show $\left\{\frac{3}{n+5}\right\}$ and $\left\{\frac{n}{n^2+1}\right\}$ are decreasing.

a) NTS: $\frac{3}{n+5} > \frac{3}{(n+1)+5} \Rightarrow \frac{3}{n+5} > \frac{3}{(n+1)+5} = \frac{3}{n+6}$

or $n+5 < n+6 \Rightarrow \frac{1}{n+6} < \frac{1}{n+5} \Rightarrow \frac{3}{(n+1)+5} < \frac{3}{n+5}$ ✓

b) <u>Method 1</u>: $\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$ <u>WTS</u>

$\Leftrightarrow (n+1)(n^2+1) < n(n+1)^2+n$

$\Leftrightarrow n^3+n^2+n+1 < n^3+2n^2+2n$

$\Leftrightarrow 1 < n^2+n$ true for all $n \geq 1 \Rightarrow a_{n+1} < a_n$

<u>Method 2</u>: $f(x) = \frac{x}{x^2+1} \Rightarrow f'(x) = \frac{1-x^2}{(x^2+1)^2} < 0$ for $x^2 > 1$

$\Rightarrow f(x)$ decreasing on $(1, \infty)$

$\Rightarrow f(n) > f(n+1) \Rightarrow \{a_n\}$ decreasing.

A sequence is <u>bounded above</u> if $\exists M \in \mathbb{R}$ s.t. $a_n \leq M$ for all $n \geq 1$

is <u>bounded below</u> if $\exists m \in \mathbb{R}$ s.t. $a_n \geq m$ for all $n \geq 1$

<u>Theorem</u>: Every bounded monotone sequence is convergent.

Ex) Let $\{a_n\}$ be defined as $a_1 = 2$
$a_{n+1} = \frac{1}{2}(a_n + 6)$ $n = 1, 2, 3, \ldots$

determine convergence or divergence.

Solution: Prove by induction. Show increasing

① Base Case: $a_2 = \frac{1}{2}(a_1 + 6) = \frac{1}{2}(2+6) = 4 > a_1 = 2$ ✓

② Assumption: $n = k$ is true ie) $a_{k+1} > a_k$

③ Prove for $n = k+1$

$$a_{k+1} > a_k$$
$$a_{k+1} + 6 > a_k + 6$$
$$\frac{1}{2}(a_{k+1} + 6) > \frac{1}{2}(a_k + 6)$$
$$a_{k+2} > a_{k+1}$$ ✓ ⇒ increasing by induction.

Show $\{a_n\}$ is bounded. We will show $a_n < 6$ for all n
(Lower bound is 2 since a_n increasing)

Again, by induction:

① Obviously $a_1 < 6 \Rightarrow a_1 = 2 < 6$

② Assume true for $n = k \Rightarrow a_k < 6$

③ Prove for $n = k+1$

$$a_k < 6 \Rightarrow a_k + 6 < 12$$
$$\frac{1}{2}(a_k + 6) < \frac{1}{2}12$$
$$a_{k+1} < 6$$ ✓

So by Thm, seq is convergent.

BUT, we do not know its Limit!! Thm does not establish this.

Call the limit L, ie. $\lim_{n\to\infty} a_n = L$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n + 6) = \frac{1}{2}\left(\left(\lim_{n\to\infty} a_n\right) + 6\right)$$
$$= \frac{1}{2}(L + 6)$$

Since $a_n \to L$ as $n \to \infty$, $a_{n+1} \to L$ as well

$$\Rightarrow L = \frac{1}{2}(L+6)$$
$$\frac{1}{2}L = 3 \Rightarrow L = 6$$ ✓

Section 8.2 - Series

When we add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get the expression

$$a_1 + a_2 + \dots + a_n + \dots = \sum_{n=1}^{\infty} a_n$$

which is called an <u>infinite</u> <u>series</u>. The question is, does this expression make sense?

Examples: $1+2+3+4+5+\dots+n+\dots$: Cumulative/partial sums $\{1, 3, 6, 10, 15, 21, \dots \frac{n(n+1)}{2}\}$

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots+\frac{1}{2^n}+\dots$: partial sum $\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \dots\} = 1-\frac{1}{2^n}$

Definition: Partial sums: determine a sequence

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$S_4 = a_1 + a_2 + a_3 + a_4$

$\vdots$

$S_n = a_1 + a_2 + a_3 + \dots + a_n = \sum_{i=1}^{n} a_i$

Form the sequence $\{S_n\} = \{S_1, S_2, S_3, S_4, S_5, \dots, S_n, \dots\}$

The sequence <u>may</u> or <u>may not</u> have a limit.

Definition: Given $\sum_{n=1}^{\infty} a_n$ and partial sum $S_n = \sum_{i=1}^{\infty} a_i$, if $\{S_n\}$ is convergent and $\lim_{n\to\infty} S_n = S$ exists and is a real number $\Rightarrow \sum_{n=1}^{\infty} a_n = S$ and we say $\sum_{n=1}^{\infty} a_n$ is <u>convergent</u>.

Otherwise, the sum is <u>divergent</u>, thus <u>the sum of the infinite series is the limit of n^{th} partial sums</u>

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

Example: Geometric Series

$$a + ar + ar^2 + ar^3 + \dots + ar^{n-1} + \dots = \sum_{n=1}^{\infty} ar^{n-1} \quad a \neq 0$$

where a = constand, r = common ratio. So

$$S_n = a + \cancel{ar} + \cancel{ar^2} + \cancel{\dots} + ar^{n-1}$$

$$\Rightarrow - \quad rS_n = \cancel{ar} + \cancel{ar^2} + \cancel{\dots} + ar^{n-1} + ar^n$$

$$S_n - rS_n = a - ar^n \Rightarrow (1-r)S_n = a(1-r^n) \Rightarrow S_n = \frac{a(1-r^n)}{(1-r)}$$

partial sum

$$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}\left(1 - \lim_{n\to\infty} r^n\right)$$

Recall that a geometric sequence is only convergent for $-1 < r \leq 1$

and also note $r=1$ gives $\frac{0}{0}$ (if $r=1 \Rightarrow S_n = na \to \infty$ for $n \to \infty$)

$$\text{So, } \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \frac{a}{1-r} \quad \text{for } |r| < 1 \text{ or } -1 < r < 1$$

Ex) $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \dots$

Ex) $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$

Ex) $\sum_{n=0}^{\infty} x^n$ for $|x| < 1$

Ex) Show $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges. Find its sum.

<u>Solution:</u> $S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right)$ by partial fractions

$\Rightarrow S_n = \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$

(telescoping sum)

$\Rightarrow S_n = 1 - \frac{1}{n+1}$, then

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1$ convergent with sum equal to 1

Show $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

<u>Solution:</u>

$S_1 = 1$

$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$

$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3}+\frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4}+\frac{1}{4}\right) = 2$

$S_8 = 1 + \frac{1}{2} + \left(\frac{1}{3}+\frac{1}{4}\right) + \left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)$

$> 1 + \frac{1}{2} + \left(\frac{1}{4}+\frac{1}{4}\right) + \left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)$

$= 1 + \frac{3}{2} = \frac{5}{2}$

$S_{16} > 1 + \frac{4}{2} = 3$

$\Rightarrow S_{2^n} > 1 + \frac{n}{2} \Rightarrow S_{2^n} \to \infty$ as $n \to \infty$

$\Rightarrow \{S_n\}$ diverges $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

<u>Theorem</u>* If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

The <u>converse</u> is <u>not</u> true!!!

Ex) $\sum_{n=1}^{\infty} \frac{1}{n} \Rightarrow \lim_{n\to\infty} \frac{1}{n} = 0$ but

$\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

<u>Test for divergence</u> If $\lim_{n\to\infty} a_n$ does not exist or

if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

If $\lim_{n\to\infty} a_n = 0$ this tells us <u>nothing</u>!! It (Series) may converge or diverge

<u>Properties</u>: If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent, then

(i) $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$ for a constant c

(ii) $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

and the result being convergent as well.

<u>Example</u>: $\sum_{n=1}^{\infty} \left(\frac{1}{(n+1)n} + \frac{1}{2^n} \right)$

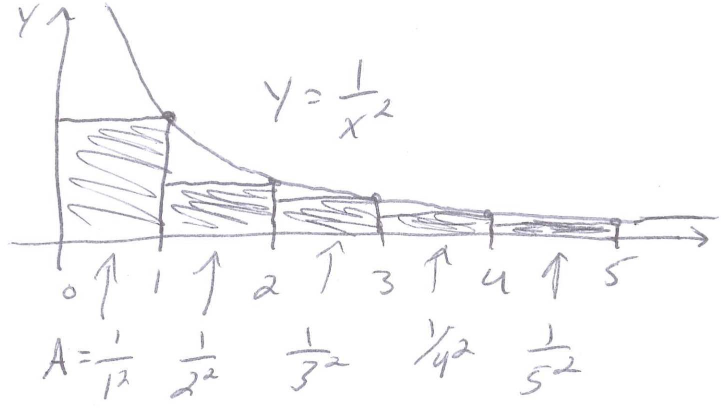
<u>Note</u>: a finite number of terms <u>does not</u> affect convergence

<u>Reindexing Series</u>. Write $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ starting at

$n=0$
$n=5$
$n=-4$

Section 8.3 - Integral and Comparison Tests

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$



$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} + \dots$$

is the area of each box for a Riemann sum added up.

The area of the rectangles is less than $\int_1^{\infty} \frac{1}{x^2}\,dx$ for $x \geq 1$

ie: $\sum_{n=1}^{\infty} \frac{1}{n^2} \leq \int_1^{\infty} \frac{1}{x^2}\,dx$, so, if we know the integral, we can determine convergence.

Integral Test: Suppose f is continuous, positive, and decreasing on $[1,\infty)$ and $a_n = f(n)$

then,
(i) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent

(ii) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent

Examples: ① $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

② $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\quad p>1$, $\; p \leq 1$

③ $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

④ $\sum_{n=1}^{\infty} n e^{-n^2}$

Comparison Tests

Comparison Test: Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

(i) If $\sum b_n$ converges and $a_n \le b_n$ for all n, then $\sum a_n$ converges

(ii) If $\sum b_n$ diverges and $a_n \ge b_n$ for all n, then $\sum a_n$ diverges

Examples: $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$, $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$, $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

Comparisons:
- Geometric $|r|<1$ converge otherwise diverge
- p-series $p>1$, $p\le 1$

Cannot compare for $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ inequality is wrong way

Limit Comparison Test: Suppose $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where c is a finite number and $c > 0$ then both series converge or both series diverge

Ex) $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

$\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{n^5+5}}$

$\sum_{n=1}^{\infty} \frac{\ln(n)}{n^{3/2}}$ (note: $\ln(n) < n^c$ for $c>0$)
$n^{1/4}$ in numerator for ex.

Section 8.4 - Ratio and Root Tests

Ratio Test: ① If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ (absolutely) converges

② If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges

③ If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio test is inconclusive
i.e. we can't say anything.

Examples: $\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{3^n}$ $\qquad \sum_{n=1}^{\infty}\frac{4^n n!n!}{(2n)!}$ $\qquad \left(\frac{a_{n+1}}{a_n} = \frac{2n+2}{2n+1} \Rightarrow \text{increasing } a_n \geq a_1 = 2 \quad \text{Test for divergence}\right)$

$\sum_{n=1}^{\infty}\frac{n^n}{n!}$ $\qquad \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2 2^n}{n!}$

Root Test: ① If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ is (absolutely) convergent

② If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ is divergent

③ If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1 \Rightarrow$ inconclusive

Note: If either the Ratio Test or Root test is $L=1$, do not try the other test, as it will also give $L=1$.

Examples: $\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n \Rightarrow \sqrt[n]{|a_n|} = \frac{2n+3}{3n+2}$

$\sum_{n=1}^{\infty}\frac{2^n}{n^3}$ $\qquad \sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$

Section 8.5 - Alternating Series and Absolute Convergence

Given a series $\sum_{n=1}^{\infty} a_n$, we can consider

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

Since $\sum_{n=1}^{\infty} a_n$ may have positive and negative terms.

Defn: A series is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

Example: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$

Defn: A sequence is conditionally convergent if it is convergent, but not absolutely convergent.

Theorem: If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

Defn: An alternating series is a series whose terms alternate signs between positive and negative

Ex) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$, etc. usually have $(-1)^n$ or $(-1)^{n+1}$

We can then always write the n^{th} term as

$$a_n = (-1)^n b_n$$ where b_n is positive (or $b_n = |a_n|$)

Alternating Series Test

If $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n b_n$, $b_n > 0$ satisfies

(a) $b_{n+1} < b_n$ for all n (b_n is decreasing)

(b) $\lim_{n \to \infty} b_n = 0$

then the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n b_n$ is convergent.

Examples: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$

Strategy for testing Series

For quizzes and tests, I will not tell you which test to use.

How do we choose the right test?

(1) If a series has a form similar to $\sum_{n=1}^{\infty} ar^{n-1}$ or $\sum_{n=1}^{\infty} \frac{1}{n^p}$

a comparison test should be used.

* For p-test, choose the highest powers of numerator and denominator

* If $\sum a_n$ has negative terms use Comparison Test with $\sum_{n=1}^{\infty} |a_n|$

(2) If you immediately see that $\lim_{n \to \infty} a_n \neq 0$, use Test for Divergence

(3) If $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n b_n$, Alternating Series Test is a good bet, but remember absolute + conditional convergence.

(4) Ratio Test is good for factorials and n^n forms

(5) Root Test is good for $\sum a_n$ that have $(b_n)^n$ terms

(6) Assuming hypotheses for the Integral Test hold, if $a_n = f(n)$ where you can compute $\int_1^{\infty} f(x)\,dx$, Integral Test is an obvious candidate.

Examples

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$ 2

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ 1

(iii) $\displaystyle\sum_{n=1}^{\infty} n e^{-n^2}$ or $\displaystyle\sum_{n=1} \frac{1}{n^2+9}$ 6

(iv) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$ 3

(v) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ 4

(vi) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+3^n}$ 1

(vii) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^n 2^n}{(n!)^n}$ 4 or 5

Section 8.6 - Power Series

A **power series** has the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

where c_n are constants. A power series may converge for only *some* values of x. A power series is a function of x, (a polynomial expression)

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

More generally we have a power series centered at a constant a

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n .$$

Ex) For which values of x does $\sum_{n=0}^{\infty} n! x^n$ converge?

Ex) For which values of x does $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

We generally apply the Ratio Test. Remember, the Ratio Test converges *only* for limits that are less than 1.

Note: we must also check *endpoints*!

Given a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, there are 3 possibilities

(i) Series *only* converges for $x = a$

(ii) Series converges for all x

(iii) $\exists$ a positive number R s.t. the series converges for $|x-a| < R$

The number R is called the radius of convergence

Case (i) above: $R=0$, only 1 point
Case (ii) above: $R=\infty$, all points

Anything can happen at the endpoints since Ratio Test is inconclusive for $L=1$.

The interval of convergence is the set of all values of x s.t. the series converges.

Chart to summarize

Name	Series	Radius of Conv. "R"	Interval of conv.
Geometric series	$\sum_{n=0}^{\infty} x^n \quad \lvert x\rvert<1$	$R=1$	$(-1,1)$
Example 1	$\sum_{n=0}^{\infty} n!x^n$	$R=0$	$\{0\}$
Example 2	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R=1$	$[2,4)$
Example 3	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$R=\infty$	$(-\infty,\infty)$

More Examples

(i) $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$ $R=\frac{1}{3}$ $\left(-\frac{1}{3},\frac{1}{3}\right]$

(iii) $\sum_{n=0}^{\infty} 2^n(x-3)^n$

(ii) $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$ $R=3$ $(-5,1)$

(iv) $\sum_{n=1}^{\infty} n^n x^n$

Representation of functions as power series

Express $\dfrac{1}{1+x^2}$ as a power series and find interval of convergence

Do so for $\dfrac{1}{x+2}$ and $\dfrac{x^3}{x+2}$

Differentiation / Integration

Theorem: If $\sum_{n=0}^{\infty} c_n(x-a)^n$ is a power series with $R>0$, then the function $f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable (and continuous) on $(a-R, a+R)$ and

$$f'(x)=\sum_{n=1}^{\infty} nc_n(x-a)^{n-1} \quad \text{and} \quad \int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence for $f'(x)$ and $\int f(x)\,dx$ are the same as $f(x)$.

Note: This is true for power series, but not in general. the interchange of $\sum_{n=1}^{\infty}$ with $\int$ or $\dfrac{d}{dx}$ is more complicated than we can discuss in this course.

Ex) Express $\dfrac{1}{(1\mp x)^2}$ as a power series

Express $-\ln(x-1)$ as a power series

Express $\arctan(x)$ as a power series

Section 8.7 - Taylor Polynomials

Consider the power series $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$

We saw last time, we can take derivatives of $f(x)$. At a point "a"

$\Rightarrow f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$

$\Rightarrow f(a) = c_0$ $\quad f''(a) = 2c_2$ $\quad f^{(4)}(a) = 2\cdot 3\cdot 4 c_4$

$f'(a) = c_1$ $\quad f'''(a) = 2\cdot 3\cdot c_3$ $\quad f^{(5)}(a) = 2\cdot 3\cdot 4\cdot 5 c_5$ etc.

$\Longrightarrow$ In general $f^{(n)}(a) = n!\, c_n \Rightarrow c_n = \dfrac{f^{(n)}(a)}{n!}$

$$\Longrightarrow \sum_{n=0}^{\infty} c_n (x-a)^n = f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

This is what is called a Taylor Series (centered at a) (or Taylor "infinite" Polynomial)

If $a=0$ in the above formula, it is called a MacLaurin Series.

Find the MacLaurin series for $f(x) = e^x$

Find the MacLaurin series for $f(x)=e^x$

~~Use~~ Denote $T_n(x)$ as the <u>n^{th} degree</u> Taylor Polynomial

For $f(x) = e^x$, $\quad T_1(x) = 1 + x$ $\quad T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$T_2(x) = 1 + x + \frac{x^2}{2!}$ $\quad$ etc.

If $f(x)$ is the sum of its own Taylor Series, then

$f(x) = \lim_{n\to\infty} T_n(x)$, i.e. limit of partial sums.

You can use $T_n(x)$ to approximate functions.

Approximate e^1 using $T_5(x)$, $n=5$

$$T_n(x) = T_5(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

$$= \frac{163}{60} \approx 2.71667$$

Find the n^{th} Taylor Polynomial for $\ln(x)$

Idea: Compute derivative on $\ln(x)$, find $f^{(n)}(x)$

$$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n}$$

If we stop at $T_n(x)$, then $R_n(x)$ is the remainder.

and $f(x) = T_n(x) + R_n(x) \Rightarrow R_n(x) = f(x) - T_n(x)$

It must be true that $\lim_{n\to\infty} R_n(x) = 0$

Since $\lim_{n\to\infty} T_n(x) = f(x)$.

Thm: (Taylor's Remainder)

If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq \delta$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \leq \delta$$

Gives a bound for remainders.

Note: $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for every real number x.

Binomial Series

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

, the $\binom{k}{n} = \frac{k(k-1)(k-2)\ldots(k-n+1)}{n!}$ are binomial coeffs.

Section 8.8 - Taylor Series

Ex) Evaluate $\int e^{-x^2}dx$ as a ~~pow~~ Taylor Series

Ex) Find Taylor Series for $e^x\cos(x)$

Ex) Find Tayor Series for $\sin(x^2)$

Examples

(i) $\ln(\sqrt{x})$

(ii) xe^{-x}

(iii) $\int x\cos(x^3)\,dx$

Use Tayor Series to solve ODE's

(i) $y' = 2y$

(ii) $x^2y'' + xy' - 2y = 0$