MATH 009C **FINAL** **August 12, 2017**

Name: ____________________________ **Score:** _______ / 100

Student ID: ____________________________

DO NOT OPEN THE EXAM UNTIL YOU ARE TOLD TO DO SO

	1	2	3	4	5	6	7	8	9	Total
✓										200
Score										
Pts. Possible	25	25	25	25	25	25	25	25	25	210

INSTRUCTIONS FOR STUDENTS

- Questions are on both sides of the paper. This is an 9 question exam.
- Students have 2 hours to complete the exam.
- The test will be out of **200** points (8 questions). You may attempt a 9^{th} question, which will have a maximum of 10 possible points. The highest possible score is therefore **210** points.
- In the above table, the row with the ✓, is for you to keep track of the problems you are attempting/completing.
- You may complete parts of problems, as partial credit will be given based on correctness, completeness, and ideas that are leading to the correct solutions.
- **PLEASE SHOW ALL WORK**. Any unjustified claims will receive no credit. This means you need to state which test you are using for series questions! Clearly box your final answer.
- No notes, textbooks, phones, calculators, etc. are allowed for the exam.
- The back of the test can be used for scratch work.

GOOD LUCK!

FORMULAS:

Common Taylor Series	Common Taylor Series
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, for all $\lvert x\rvert < 1$	$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, for all $x \in \mathbb{R}$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, for all $x \in \mathbb{R}$	$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, for all $x \in \mathbb{R}$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$, for $x \in (-1, 1]$	$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$, for $\lvert x\rvert \leq 1$
$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, for $\lvert x-a\rvert < R$	$(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n$, for $\lvert x\rvert < 1$

1) (10 pts.) (a) Determine whether the sequence converges or diverges:

$$a_n = \frac{\cos^2(n)}{2^n}.$$

(15 pts.) (b) Determine whether the sequence converges or diverges:

$$a_n = n \sin\left(\frac{1}{n}\right).$$

Solution:

(a) By using inequalities, we have $|\cos^2(n)| \le 1$, and it is easy to see that $\cos^2(n) \ge 0$ and $2^n > 0$, so we have

$$0 \le \frac{\cos^2(n)}{2^n} \le \frac{1}{2^n}.$$

So by Squeeze Theorem, using

$$\lim_{n\to\infty} \frac{1}{2^n} = 0,$$

the sequence a_n converges.

(b) Here, we rewrite the sequence using the same limit trick we've done many times with the logarithm limits, and a substitution $t = 1/n$

$$\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{1/n} = \lim_{t\to 0^+} \frac{\sin(t)}{t} = 1,$$

where the limits change due to the substitution. Since the limit is equal to 1, the sequence converges.

2) (10 pts.) Determine whether the series is convergent or divergent:

$$\sum_{n=1}^{\infty} \arctan(n).$$

(15 pts.) (b) Determine whether the series is convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}.$$

Solution:

(a) Use the Test for Divergence:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \arctan(n) = \frac{\pi}{2} \neq 0.$$

Since the limit is not equal to zero, the series diverges.

(b) The series can be divided into 2 geometric series:

$$\sum_{n=1}^{\infty} \frac{1+2^n}{3^n} = \sum_{n=1}^{\infty} \frac{1}{3^n} + \frac{2^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n,$$

and since both geometric series converge because $|1/3| < 1$ and $|2/3| < 1$, so the original series is convergent.

3) (25 pts.) Determine whether the series is convergent or divergent

$$\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}.$$

Solution:

Use the Integral Test. Compare with $f(x) = \dfrac{1}{x \ln(x)}$ (This should look familiar from the quiz!)

- $f(x)$ is continuous on $[2, \infty)$ as its only undefined at $x = 0$ and $x = 1$.
- $f(x)$ is positive as $x > 0$ and $\ln(x) > 0$ for $x > 1$.
- $f'(x) = -\dfrac{1 + \ln(x)}{(x \ln(x))^2} < 0$ for $x \geq 2 \quad \Rightarrow \quad f(x)$ is decreasing.

Then by computation

$$\begin{aligned} \int_2^{\infty} \frac{1}{x \ln(x)} \, dx &= \lim_{t \to \infty} \ln(\ln(x)) \,\Big|_2^t \\ &= \lim_{t \to \infty} \ln(\ln(t)) - \ln(\ln(2)) \\ &= \infty. \end{aligned}$$

So the series is divergent by Integral Test.

4) (25 pts.) Determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right).$$

Solution:

Use the Limit Comparison Test with the harmonic series, and the trick from Problem 1(b):

$$\begin{aligned}\lim_{n\to\infty} \frac{\tan(1/n)}{1/n} = \lim_{t\to 0^+} \frac{\tan(t)}{t} &= \lim_{t\to 0^+} \left[\left(\frac{1}{t}\right)\left(\frac{\sin(t)}{\cos(t)}\right)\right] \\ &= \left(\lim_{t\to 0^+} \frac{\sin(t)}{t}\right)\left(\lim_{t\to 0^+} \frac{1}{\cos(t)}\right) \\ &= 1\cdot 1 = 1 > 0.\end{aligned}$$

Therefore, since the limit is positive and finite, and the harmonic series diverges, the original series diverges by Limit Comparison Test.

5) (15 pts.) (a) Determine whether the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}.$$

(10 pts.) (b) Determine whether the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}.$$

Solution:

(a) Use the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{(n+1)!}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n!} \\
&= \lim_{n\to\infty} \frac{(n+1)n!}{n!} \cdot \frac{e^{n^2}}{e^{n^2+2n+1}} \\
&= \lim_{n\to\infty} \frac{n+1}{e^{2n+1}} = 0 < 1.
\end{aligned}$$

So the series is absolutely convergent by the Ratio Test.

(b) Use the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)^{n^2}\right)^{1/n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n} = e > 1.$$

So the series diverges by the Root Test.

6) (25 pts.) Determine whether the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n^2-1}}.$$

Solution:

(1) Check absolute convergence: Use Limit Comparison with harmonic series. Take absolute value:

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\sqrt{n^2-1}} \right| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}.$$

Now do comparison

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} \cdot \frac{n}{1} \stackrel{L'Hop}{=} \lim_{n\to\infty} \frac{n}{\sqrt{n^2}} = 1 > 0.$$

Since the harmonic series is divergent, the above series diverges. Therefore, the series *does not* converge absolutely.

(2) Check Alternating Series Test:

(a) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n^2-1}} = 0$

(b) Now we must show that b_n is decreasing. Take derivative, or:

$$\sqrt{n^2-1} < \sqrt{(n+1)^2-1} \quad \Rightarrow \quad \frac{1}{\sqrt{(n+1)^2-1}} < \frac{1}{\sqrt{n^2-1}} \quad \Rightarrow \quad b_{n+1} < b_n$$

So the b_n are decreasing. Therefore, the series converges by Alternating Series Test. Since the series did not converge absolutely, the series is **conditionally convergent**.

7) (25 pts.) Find the radius of convergence and interval of convergence for the following power series (This is known as the *Bessel function of order 1*):

$$\sum_{n=0}^{\infty} \frac{(-1)^n\ x^{2n+1}}{n!\ (n+1)!\ 2^{2n+1}}$$

Solution:

Use the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(n+1)!(n+2)!2^{2n+3}}\cdot\frac{n!(n+1)!2^{2n+1}}{x^{2n+1}}\right| \\
&= \lim_{n\to\infty}\left|\frac{x^{2n+3}}{x^{2n+1}}\right|\cdot\frac{2^{2n+1}}{2^{2n+3}}\cdot\frac{n!(n+1)!}{(n+2)(n+1)n!(n+1)!} \\
&= \frac{|x|^2}{2^2}\lim_{n\to\infty}\frac{1}{(n+1)(n+2)} = 0 \qquad \text{for all } x.
\end{aligned}$$

The limit being equal to 0 implies that the radius of convergence is $R = \infty$, and the interval of convergence is $(-\infty, \infty)$.

8) Find the sum of the following series:

(5 pts.) (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)! 4^{2n+1}}$

(5 pts.) (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)! 6^{2n}}$

(15 pts.) (c) $\displaystyle\sum_{n=1}^{\infty} (-1^n) \frac{x^{4n}}{n!}$

Solution:

(a) Use Formula 1, Column 2 from Table:

$$\sum_{n=1}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)! 4^{2n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi}{4}\right)^{2n+1} = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

(b) Use Formula 2, Column 2 from Table:

$$\sum_{n=1}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)! 6^{2n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\pi}{6}\right)^{2n} = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

(c) Use Formula 2, Column 1 from Table:

$$\sum_{n=1}^{\infty} (-1^n) \frac{x^{4n}}{n!} = \sum_{n=1}^{\infty} \frac{(-x^4)^n}{n!} = e^{-x^4}$$

9) (15 pts.) (a) Compute the following integral using Taylor series. (*Hint: Be careful about the $n = 0$ term, you can't have 0/0.*)

$$\int \frac{e^x}{x}\, dx$$

(10 pts.) (b) Find the Taylor series centered at $a = 0$ for the function

$$f(x) = 2xe^{x^2}$$

Solution:

(a) We know from the table on the front (line 1 below)

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
\frac{1}{x}e^x &= \sum_{n=0}^{\infty} \frac{x^{n-1}}{n!} \\
&= \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} \\
\int \frac{e^x}{x}\, dx &= \int \frac{1}{x}\, dx + \sum_{n=1}^{\infty} \int \frac{x^{n-1}}{n!}\, dx \\
&= \ln|x| + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!} + C
\end{aligned}$$

(b) There are two ways to do this. First use a substitution of x^2 into the formula, then multiply by $2x$:

$$\begin{aligned}
e^{x^2} &= \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \\
2xe^{x^2} &= \sum_{n=0}^{\infty} 2x\frac{x^{2n}}{n!} = \sum_{n=0}^{\infty} 2\frac{x^{2n+1}}{n!} = 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}
\end{aligned}$$

Or take a derivative of the first equation

$$2xe^{x^2} = \frac{d}{dx}e^{x^2} = \sum_{n=1}^{\infty} 2n\frac{x^{2n-1}}{n!} = 2\sum_{n=1}^{\infty} \frac{x^{2n-1}}{(n-1)!} = 2\sum_{n=0}^{\infty} \frac{x^{2(n+1)-1}}{n!} = 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$$

THIS PAGE IS LEFT BLANK FOR ANY SCRATCH WORK

END OF TEST