MATH 009C **MIDTERM** **July 13, 2017**

Name: ____________________________ **Score:** _______ / 100

Student ID: ____________________________

DO NOT OPEN THE EXAM UNTIL YOU ARE TOLD TO DO SO

	1	2	3	4	5	Total
✓						
Score						
Pts. Possible	25	25	25	25	25	110

INSTRUCTIONS FOR STUDENTS

- You can use both sides of the paper for your solution. This is an **4** question exam.
- Students have **50** minutes to complete the exam.
- The test will be out of **100** points (4 questions). You may attempt a 5^{th} question, which will have a maximum of 10 possible points. The highest possible score is therefore **110** points.
- In the above table, the row with the ✓, is for you to keep track of the problems you are attempting/completing.
- You may complete parts of problems, as partial credit will be given based on correctness, completeness, and ideas that are leading to the correct solutions.
- **PLEASE SHOW ALL WORK**. Any unjustified claims will receive no credit.
- No notes, textbooks, phones, calculators, etc. are allowed for the exam.
- The last page of the test can be used for scratch work.

GOOD LUCK!

1) (25 pts.) Determine if the integral is convergent or divergent:

$$\int_1^{\infty} \frac{x}{x^3+1}\, dx.$$

Solution:

Use the Direct Comparison Test (or Limit Comparison Test) for Integrals.

$$x^3 < x^3+1 \quad \Rightarrow \quad \frac{1}{x^3+1} < \frac{1}{x^3} \quad \Rightarrow \quad \frac{x}{x^3+1} < \frac{x}{x^3} = \frac{1}{x^2} \qquad \text{for all } x > 1.$$

So then by computation,

$$0 < \int_1^{\infty} \frac{x}{x^3+1}\, dx < \int_1^{\infty} \frac{1}{x^2}\, dx = \lim_{t\to\infty} -\frac{1}{t} + 1 = 1 < \infty.$$

So by Direct Comparison Test, the integral converges.

2) (20 pts.) (a) Eliminate the parameter in the for the following parametric equation

$$x = 4\cos(t) + 2 \qquad y = 2\sin(t) + 1$$

(5 pts.) (b) Identify the type of graph from your result in part (a).

Solution:

(a) Rewrite x and y in terms of $\cos^2(t)$ and $\sin^2(t)$:

$$x = 4\cos(t) + 2$$
$$(x-2)^2 = 16\cos^2(t)$$
$$\frac{(x-2)^2}{16} = \cos^2(t)$$

and

$$y = 2\sin(t) + 1$$
$$(y-1)^2 = 4\sin^2(t)$$
$$\frac{(y-1)^2}{4} = \sin^2(t)$$

By using the trig identity $\sin^2(t) + \cos^2(t) = 1$:

$$\frac{(x-2)^2}{16} + \frac{(y-1)^2}{4} = 1$$

(b) The graph is an ellipse centered at $(2, 1)$ with major axis $a = 4$, and minor axis $b = 2$.

3) Consider the parametric defined as: $x = t - e^t, y = t + e^{-t}$

(10 pts.) (a) Compute $\dfrac{dy}{dx}$.

(10 pts.) (b) Compute $\dfrac{d^2y}{dx^2}$.

(5 pts.) (c) For which values of t is the curve concave upward?

Solution:

(a) Use the formula for the derivative:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 - e^{-t}}{1 - e^t} = \frac{1}{1 - e^t}\left(1 - \frac{1}{e^t}\right) = \frac{1}{1 - e^t}\frac{-(1 - e^t)}{e^t} = -e^{-t}$$

(b) Use the formula for the second derivative:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(-e^{-t})}{1 - e^t} = \frac{e^{-t}}{1 - e^t}$$

(c) Since $e^{-t} > 0$ for all t, we only need to check the sign of the denominator. So then, $1 - e^t > 0$ when $e^t < 1$. Therefore, the curve is concave upward for $e^t < 1$ (or $t < \ln(1) = 0$).

4) (10 pts.) (a) Consider the polar curve $r = 1/\theta$. Find the slope of the tangent line to the curve at the point $\theta = \pi$.

(15 pts.) (b) Consider the polar curve $r = 1 + \sin(\theta)$. Find the θ where the tangent line is vertical for $0 \le \theta < 2\pi$. State which value of θ give $\dfrac{dy}{dx} = \dfrac{0}{0}$ (you do not need to compute the limit).

Solution:

(a) Use the formula for the derivative:

$$\begin{aligned}\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)} &= \frac{-\frac{1}{\theta^2}\sin(\theta) + \frac{1}{\theta}\cos(\theta)}{-\frac{1}{\theta^2}\cos(\theta) - \frac{1}{\theta}\sin(\theta)} \\ &= \frac{-\frac{1}{\theta^2}\sin(\theta) + \frac{1}{\theta}\cos(\theta)}{-\frac{1}{\theta^2}\cos(\theta) - \frac{1}{\theta}\sin(\theta)} \cdot \frac{\theta^2}{\theta^2} \\ &= \frac{-\sin(\theta) + \theta\cos(\theta)}{-\cos(\theta) - \theta\sin(\theta)} \\ &\overset{\theta=\pi}{=} -\pi\end{aligned}$$

(b) Use the formula for the derivative:

$$\begin{aligned}\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)} &= \frac{\cos(\theta)\sin(\theta) + (1+\sin(\theta))\cos(\theta)}{\cos^2(\theta) - (1+\sin(\theta))\sin(\theta)} \\ &= \frac{\cos(\theta)\sin(\theta) + \cos(\theta) + \sin(\theta)\cos(\theta)}{\cos^2(\theta) - \sin(\theta) - \sin^2(\theta)} \\ &= \frac{\cos(\theta)(2\sin(\theta)+1)}{-(2\sin^2(\theta) + \sin(\theta) - 1)} \\ &= \frac{\cos(\theta)(2\sin(\theta)+1)}{-(2\sin(\theta)-1)(\sin(\theta)+1)}\end{aligned}$$

We only need to find where the denominator is zero, we only solve two equations:

$$\begin{aligned} 2\sin(\theta) - 1 &= 0 & \sin(\theta) + 1 &= 0 \\ \sin(\theta) &= \frac{1}{2} & \sin(\theta) &= -1 \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} & \theta &= \frac{3\pi}{2}\end{aligned}$$

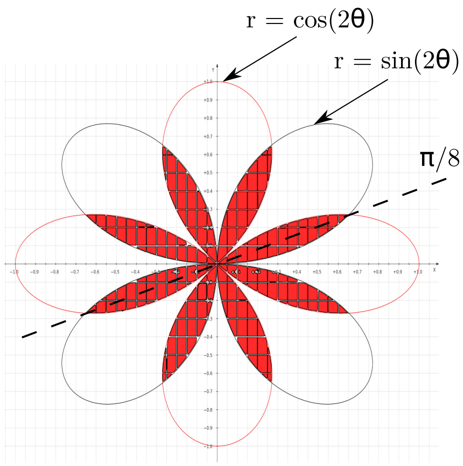
The value of θ such that $\dfrac{dy}{dx} = \dfrac{0}{0}$, is $\dfrac{3\pi}{2}$, since $\theta = \dfrac{3\pi}{2}$ gives $\cos\left(\dfrac{3\pi}{2}\right) = 0$.

5) (25 pts.) Find the area of the region that lies inside the polar roses (the region is shaded in the labeled plot below).

$$r = \cos(2\theta)$$
$$r = \sin(2\theta)$$

Hint 1: Use the symmetry at $\frac{\pi}{8}$ *to simplify the integral. How many pieces do you really have?*
Hint 2: Identities that may be helpful: 1) $\sin^2(2\theta) = \frac{1}{2} - \frac{1}{2}\cos(4\theta)$, *2)* $\cos^2(2\theta) = \frac{1}{2} + \frac{1}{2}\cos(4\theta)$.

Solution:



Find the intersection point first (the picture already gives you the value). We have to solve the equation:

$$\sin(2\theta) = \cos(2\theta)$$
$$\tan(2\theta) = 1$$
$$2\theta = \frac{\pi}{4}, \frac{5\pi}{4}$$
$$\theta = \frac{\pi}{8}, \frac{5\pi}{8}$$

Using hint 1, we have 8 petals, but by cleverly choosing the second function, which starts at the origin at $\theta = 0$, from $\theta = 0$ to $\theta = \pi/8$, we get half of the first petal. Since the polar rose is symmetric, really have 16 half petals. Now we compute and use hint 2:

$$\begin{aligned} A &= 8 \cdot 2 \int_0^{\pi/8} \frac{1}{2} r^2 \, d\theta \\ &= 16 \int_0^{\pi/8} \frac{1}{2} \sin^2(2\theta) \, d\theta \\ &= 8 \int_0^{\pi/8} \left(\frac{1}{2} - \frac{1}{2}\cos(4\theta) \right) \, d\theta \\ &= 4 \left(\theta - \frac{1}{4}\sin(4\theta) \right) \Big|_0^{\pi/8} \\ &= \frac{\pi}{2} - 1 \end{aligned}$$

THIS PAGE IS LEFT BLANK FOR ANY SCRATCH WORK

END OF TEST