# Section 8.5 - Alternating Series and Absolute Convergence

Given a series $\sum_{n=1}^{\infty} a_n$, we can consider

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

Since $\sum_{n=1}^{\infty} a_n$ may have positive and negative terms.

**Defn:** A series is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

**Example:** $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$

**Defn:** A sequence is conditionally convergent if it is convergent, but not absolutely convergent.

**Theorem:** If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

**Defn:** An alternating series is a series whose terms alternate signs between positive and negative

Ex) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$, etc. usually have $(-1)^n$ or $(-1)^{n+1}$

We can then always write the $n$th term as

$$a_n = (-1)^n b_n$$ where $b_n$ is positive (or $b_n = |a_n|$)

## Alternating Series Test

If $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n b_n$, $b_n > 0$ satisfies

(a) $b_{n+1} < b_n$ for all $n$ ($b_n$ is decreasing)

(b) $\lim_{n\to\infty} b_n = 0$

then the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n b_n$ is convergent.

Examples: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$

## Strategy for testing Series

For quizzes and tests, I will not tell you which test to use.
How do we choose the right test?

① If a series has a form similar to $\sum_{n=1}^{\infty} ar^{n-1}$ or $\sum_{n=1}^{\infty} \frac{1}{n^p}$

a comparison test should be used.

- * For p-test, choose the highest powers of numerator and denominator
- * If $\sum a_n$ has negative terms use Comparison Test with $\sum_{n=1}^{\infty} |a_n|$

② If you immediately see that $\lim_{n\to\infty} a_n \neq 0$, use Test for Divergence

③ If $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n b_n$, Alternating Series Test is a good bet, but remember absolute + conditional convergence.

④ Ratio Test is good for factorials and $n^n$ forms

⑤ Root Test is good for $\sum a_n$ that have $(b_n)^n$ terms

⑥ Assuming hypotheses for the Integral Test hold, if $a_n = f(n)$ where you can compute $\int_1^{\infty} f(x)\,dx$, Integral Test is an obvious candidate.

# Examples

(i) $\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$ 2

(ii) $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ 1

(iii) $\sum_{n=1}^{\infty} n e^{-n^2}$ or $\sum_{n=1} \frac{1}{n^2+9}$ 6

(iv) $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$ 3

(v) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ 4

(vi) $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$ 1

(vii) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^n 2^n}{(n!)^n}$ 4 or 5