Section 8.6 - Power Series

A power series has the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where c_n are constants. A power series may converge for only some values of x. A power series is a function of x, (a polynomial expression)

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

More generally we have a power series centered at a constant a

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n.$$

Ex) For which values of x does $\sum_{n=0}^{\infty} n! x^n$ converge?

Ex) For which values of x does $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

We generally apply the Ratio Test. Remember, the Ratio Test converges only for limits that are less than 1.

Note: we must also check endpoints!

Given a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, there are 3 possibilities

(i) Series only converges for $x = a$

(ii) series converges for all x

(iii) $\exists$ a positive number R s.t. the series converges for $|x-a| < R$

The number R is called the <u>radius of convergence</u>

Case (i) above: $R=0$, only 1 point

Case (ii) above: $R=\infty$, all points

Anything can happen at the endpoints since Ratio Test is inconclusive for $L=1$.

The interval of convergence is the set of <u>all</u> values of x s.t. the series converges.

Chart to summarize

Name	Series	Radius of Conv. "R"	Interval of Conv.
Geometric series	$\sum_{n=0}^{\infty} x^n \quad \lvert x\rvert<1$	$R=1$	$(-1,1)$
Example 1	$\sum_{n=0}^{\infty} n!x^n$	$R=0$	$\{0\}$
Example 2	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R=1$	$[2,4)$
Example 3	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$R=\infty$	$(-\infty,\infty)$

More Examples

(i) $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$ $\quad R=\frac{1}{3}$, $\left(-\frac{1}{3},\frac{1}{3}\right]$

(iii) $\sum_{n=0}^{\infty} 2^n(x-3)^n$

(ii) $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$ $\quad R=3$, $(-5,1)$

(iv) $\sum_{n=1}^{\infty} n^n x^n$

Representation of functions as power series

Express $\frac{1}{1+x^2}$ as a power series and find interval of convergence

Do so for $\frac{1}{x+2}$ and $\frac{x^3}{x+2}$

Differentiation / Integration

Theorem: If $\sum_{n=0}^{\infty} c_n(x-a)^n$ is a power series with $R>0$, then the function $f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable (and continuous) on $(a-R, a+R)$ and

$$f'(x)=\sum_{n=1}^{\infty} n c_n (x-a)^{n-1} \quad \text{and} \quad \int f(x)\,dx = C+\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence for $f'(x)$ and $\int f(x)\,dx$ are the same as $f(x)$.

Note: This is true for power series, but not in general. The interchange of $\sum_{n=1}^{\infty}$ with $\int$ or $\frac{d}{dx}$ is more complicated than we can discuss in this course.

Ex) Express $\frac{1}{(1\mp x)^2}$ as a power series

Express $-\ln(x-1)$ as a power series

Express $\arctan(x)$ as a power series