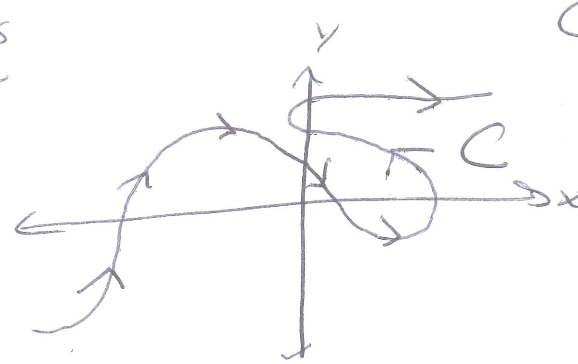
Section 9.2 - Parametric Equations



Imagine a particle moving along along a curve C shown to the right. We cannot describe the curve C by a function $y=f(x)$, as it fails the "vertical line test."

Instead, we introduce a third variable, say "t", which is called the parameter. So for a given t, we can construct two functions

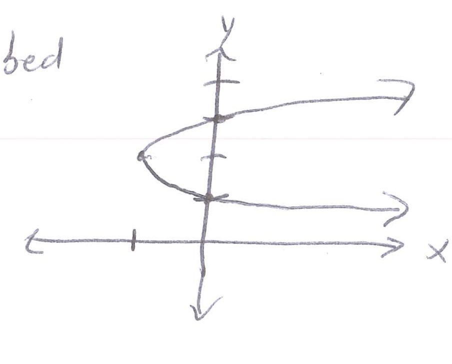
$$x = f(t) \quad \text{and} \quad y = g(t)$$

which produce the x and y coordinates of a graph. These equations are parametric equations. Each ~~(x,y)~~ (x,y) in $\mathbb{R}^2$ which can trace out C. We then call C a parametric curve

Ex) Sketch and identify the curve decribed by

$$x = t^2 - 2t$$
$$y = t + 1$$

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3



a sideways parabola

We can "eliminate the parameter" t

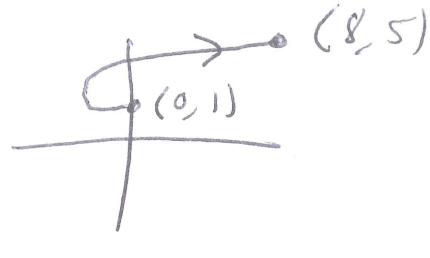
$x = t^2 - 2t$

$= (y-1)^2 - 2(y-1)$

$x = y^2 - 4y + 3 \Rightarrow$ parabola

We can also restrict values of t to get a portion of the graph.

$\begin{cases} x = t^2 - 2t \\ y = t + 1 \end{cases}$ $0 \le t \le 4$ $\Rightarrow$



So if we have $a \le t \le b$

$(f(a), g(a))$ is the initial point

$(f(b), g(b))$ is the terminal point

So the graph has a direction

Eliminate the parameter

Ex) $\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases} \Rightarrow \begin{matrix} x^2 = \cos^2(t) \\ y^2 = \sin^2(t) \end{matrix} \Rightarrow x^2 + y^2 = \cos(t) + \sin(t)$

$\boxed{x^2 + y^2 = 1}$ circle $r = 1$ center $(0,0)$

Ex) $\begin{cases} x = h + r\cos(t) \\ y = k - r\sin(t) \end{cases} \Rightarrow \begin{matrix} x - h = r\cos(t) \\ y - k = r\sin(t) \end{matrix} \Rightarrow \begin{matrix} (x-h)^2 = r^2\cos^2(t) \\ (y-k)^2 = r^2\sin^2(t) \end{matrix}$

$\Rightarrow (x-h)^2 + (y-k)^2 = r^2(\cos^2(t) + \sin^2(t))$

$\Rightarrow \boxed{(x-h)^2 + (y-k)^2 = r^2}$

Ex) $\begin{cases} x = \dfrac{1}{t^2+1} \\ y = \dfrac{t^2}{t^2+1} \end{cases} \Rightarrow xt^2 + x - 1 = 0$ or $\frac{1}{x} - 1 = t^2$ so $t = \pm\sqrt{\frac{1}{x} - 1}$

$\Rightarrow y = \dfrac{\frac{1}{x} - 1}{\frac{1}{x} - 1 + 1} = \dfrac{\frac{1}{x} - 1}{\frac{1}{x}} = \boxed{1 - x}$

So we have a line $y = 1 - x$.

Convert rectangular Cartesian to parametric

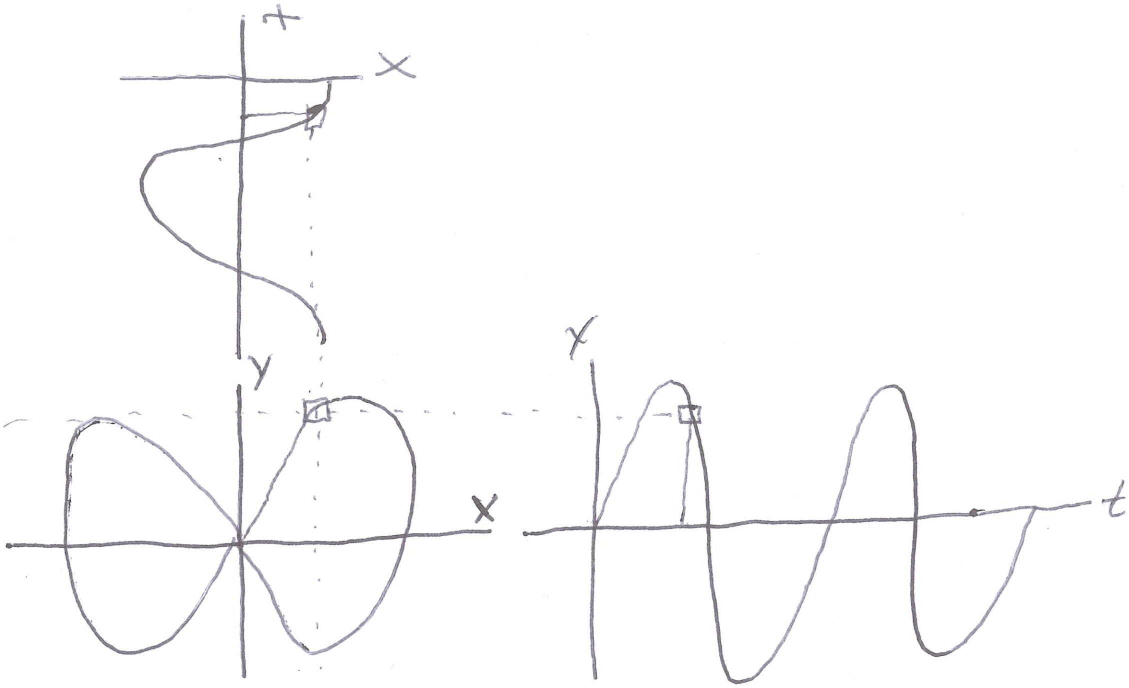
Let $y = x^2$, find f, g s.t. $x = f(t)$ and $y = g(t)$, where $\frac{dy}{dx} = t$

So $\frac{dy}{dx} = 2x \Rightarrow 2x = t \Rightarrow x = \frac{t}{2} \Rightarrow y = x^2 = \frac{t^2}{4}$

So $\boxed{\begin{matrix} x = \frac{t}{2} \\ y = \frac{t^2}{4} \end{matrix}}$

Sketching graphs

Sketch $x = \cos(t)$, $y = \sin(2t)$



Ex) $x = \sqrt{t}$ $\quad t \geq 0$ $\quad \left(\Rightarrow y = t = (\sqrt{t})^2 = x^2 \quad t \geq 0 \Rightarrow x \geq 0 \right)$

$y = t$

Ex) $x = t$ $\quad$ all t

$y = t^2$

Ex) $x = t + \frac{1}{t}$ $\quad t > 0$ $\quad$ (Find $x-y$ and $x+y$

$y = t - \frac{1}{t}$ $\quad$ $(x-y)(x+y) \Rightarrow x^2 - y^2 = 4$)