Section 9.5 - Calculus with Polar Graphs

Tangent Lines

Recall polar coordinate formulas: $x = r\cos(\theta)$ $\quad y = r\sin(\theta)$

$x = f(\theta)\cos(\theta)$ $\quad y = f(\theta)\sin(\theta)$

and $r = f(\theta)$

$$\Longrightarrow \begin{cases} \dfrac{dx}{d\theta} = f'(\theta)\cos(\theta) - f(\theta)\sin(\theta) \\[2mm] \dfrac{dy}{d\theta} = f'(\theta)\sin(\theta) + f(\theta)\cos(\theta) \end{cases}$$

or

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)$$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)$$

Then by chain rule (like parametric $\frac{dy}{dx}$)

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)}}$$

Ex) Let $r = 1 + 2\sin(\theta)$ $\quad 0 \le \theta \le 2\pi$

① Find tangent line at $\pi/4 = \theta$

② Find ~~(x,y)~~ ~~where~~ θ where there is horiz or vert. tangent lines

Solution: ①

$$\frac{dy}{dx} = \frac{2\cos(\theta)\sin(\theta) + \cos(\theta)(1+2\sin(\theta))}{2\cos^2(\theta) - \sin(\theta)(1+2\sin(\theta))}$$

$$\left.\frac{dy}{dx}\right|_{\theta = \pi/4} = -2\sqrt{2} - 1$$

$y_0 = r_0\sin(\theta_0)$

$y_0 = (1+\sqrt{2})\frac{\sqrt{2}}{2} = 1 + \frac{\sqrt{2}}{2}$

$x_0 = r_0\cos(\theta_0)$

$x_0 = (1+\sqrt{2})\frac{\sqrt{2}}{2} = 1 + \frac{\sqrt{2}}{2}$

$y - y_0 = (x - x_0)m \Rightarrow$

$m = -2\sqrt{2} - 1$

$$\boxed{y = (-2\sqrt{2}-1)\left(x - \left(1 + \frac{\sqrt{2}}{2}\right)\right) - \left(1 + \frac{\sqrt{2}}{2}\right)}$$

(b) Horiz: $\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{d\theta} = 0 \Rightarrow \cos(\theta)(4\sin(\theta)+1) = 0$

$\boxed{\theta = \pi/2, 3\pi/2}$ since $\cos(\theta) = 1$

$\boxed{\theta = \sin^{-1}(-1/4)}$ since $\sin(\theta) = -1/4$

Vert: $\frac{dx}{d\theta} = 0 \Rightarrow 2(\cos^2(\theta) - \sin^2(\theta)) - \sin(\theta) = 0$

$\cos^2\theta = 1 - \sin^2\theta$

$$2(1 - 2\sin^2(\theta)) - \sin(\theta) = 0$$

$$-4\sin^2(\theta) - \sin(\theta) + 2 = 0$$

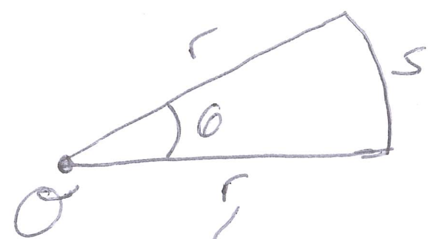
$$4\sin^2(\theta) + \sin(\theta) - 2 = 0$$

$$\sin(\theta) = \frac{-1}{2(4)} \pm \frac{\sqrt{1-4(4)(-2)}}{2(4)}$$

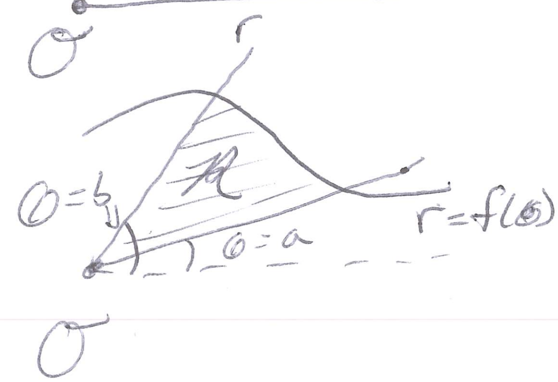
$$\sin(\theta) = -\frac{1}{8} \pm \frac{\sqrt{33}}{8}$$

$$\boxed{\theta = \sin^{-1}\left(\frac{-1 \pm \sqrt{33}}{8}\right)}$$

Areas and Lengths



Recall the area of a sector: $A = \frac{1}{2}r^2\theta$



For a general area, say the region labeled as R in the figure, then

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2 \, d\theta \quad \text{or} \quad A = \int_a^b \frac{1}{2} r^2 \, d\theta$$

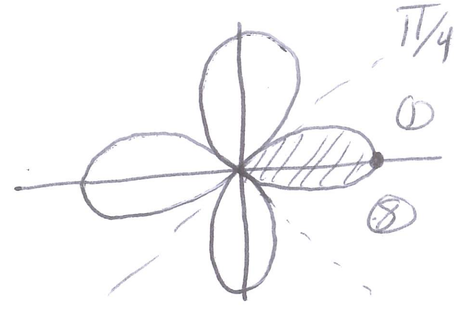
with the understanding that

$r = f(\theta)$!!

Find the area of one loop of

$r = \cos(2\theta)$

Recall the picture from last time



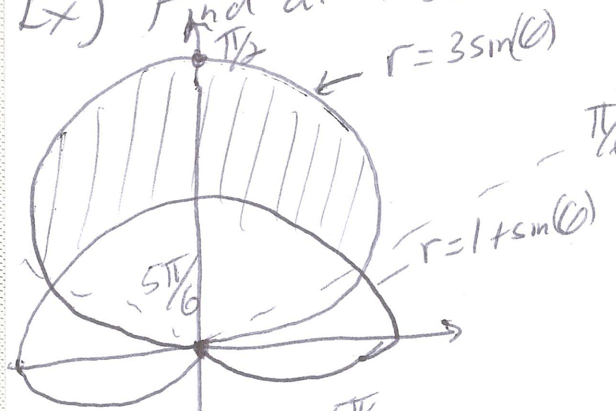
From the fact that for $\theta \in [\frac{-\pi}{4}, \frac{\pi}{4}]$ we get one petal, then

$$A = \frac{1}{2}\int_{-\pi/4}^{\pi/4} r^2 \, d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2(2\theta)\, d\theta = 2\left(\frac{1}{2}\int_0^{\pi/4} \cos^2(2\theta)\, d\theta\right)$$

$$= \int_0^{\pi/4} \frac{1}{2}(1+\cos(4\theta))\, d\theta = \frac{1}{2}\left[\theta + \frac{1}{4}\sin(4\theta)\right]\Big|_0^{\pi/4}$$

$$= \boxed{\pi/8}$$

Ex) Find area between $r = 3\sin(\theta)$ (outside cardoid) and $r = 1+\sin(\theta)$



Solution: Find intersections first

$3\sin(\theta) = 1+\sin(\theta)$

$2\sin(\theta) = 1 \Rightarrow \sin(\theta) = \frac{1}{2}$

$\theta = \pi/6, 5\pi/6$

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin(\theta))^2 d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1+\sin(\theta))^2 d\theta$$

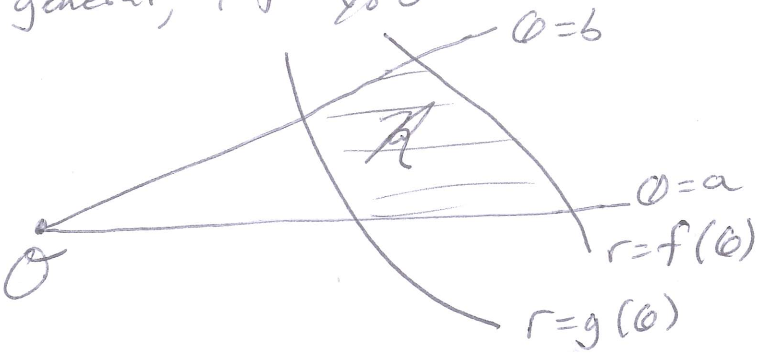
$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$

Symmetry

$$= 2 \cdot \frac{1}{2}\left[\int_{\pi/6}^{\pi/2} (8\sin^2\theta - 2\sin\theta - 1)\, d\theta\right.$$

$$\boxed{A = \pi}$$

In general, if you have the followins



$$A = \frac{1}{2}\int_a^b \left((f(\theta))^2 - (g(\theta))^2 \right) d\theta$$

Example: Find the intersection points of $r = \cos(2\theta)$, $r = \frac{1}{2}$

$$\Rightarrow \quad \cos(2\theta) = \frac{1}{2}$$

$$\cos(u) = \frac{1}{2} \Rightarrow u = 2\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

But there are 4 more!!

$$\cos(2\theta) = -\frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

Example: Practice Midterm #5

Arc Length: $r = f(\theta)$, $a \le \theta \le b$, $L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

Find arc length of $r = 1 + \sin(\theta)$ $0 \le \theta \le 2\pi$

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}$$

$$= 8$$

Use $x = \frac{\pi}{2} - y$

$$\sqrt{2}\int_{-3\pi/2}^{\pi/2} \sqrt{1+\cos(y)}\, dy$$

$$= 2\int_{-3\pi/2}^{\pi/2} |\cos(y/2)|\, dy$$

$$= 2\left[-\int_{-3\pi/2}^{-\pi} \cos(y/2)\, dy + \int_{-\pi}^{\pi/2} \cos(y/2)\, dy \right]$$