Section 8.1 - Sequences

A *sequence* is a list of numbers with a definite order

$$a_1, a_2, a_3, \dots, a_n, \dots$$

also denoted as $\{a_1, a_2, a_3, \dots\}$, $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

Ex) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$, $\left\{\frac{(-1)^n(n+1)}{3^n}\right\}$, $\left\{\cos\left(\frac{n\pi}{2}\right)\right\}_{n=0}^{\infty}$

Ex) Find a formula, a_n, for the sequence $\left\{\frac{3}{5}, \frac{-4}{25}, \frac{5}{125}, \frac{-6}{625}, \dots\right\}$

Ex) *Recursive sequences*: $f_1 = f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$ $n \geq 3$ Fibonacci sequence

Defn: A *sequence* has a limit, L, written as

(i) $\lim_{n\to\infty} a_n = L$ or (ii) $a_n \to L$ as $n \to \infty$

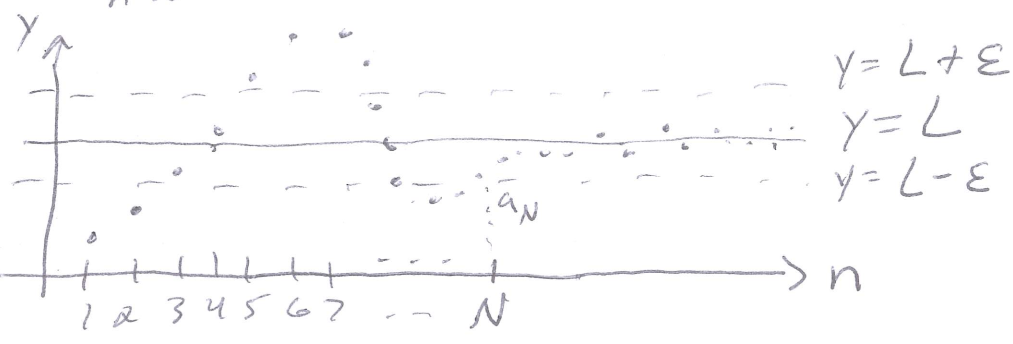
if a_n is as close to L as we like for sufficiently large n.

Rigorous: A sequence has a limit L if for every $\varepsilon > 0$ there exists an integer N s.t. for $n \geq N$,

$$|a_n - L| < \varepsilon.$$

Note: If $\lim_{n\to\infty} a_n$ exists, and is finite, we say the sequence *converges*.

If $\lim_{n\to\infty} a_n = \infty$ or DNE, then the sequence *diverges*.



<u>Theorem</u>: If $\lim_{x \to \infty} f(x) = L$, and $f(n) = a_n$, with $n \in \mathbb{Z}$, then

$$\lim_{n \to \infty} a_n = L$$

<u>Defn</u>: If $\lim_{n \to \infty} a_n = \infty \Rightarrow$ for every $M > 0$, there exists an integer N s.t. for $n \geq N$, $a_n \geq M$

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences we have:
sum rule, difference rule, product rule, quotient rule, power rule
i.e) sums of sequences converge, products converge, etc.

<u>Squeeze Theorem</u>: If $a_n \leq b_n \leq c_n$ for $n \geq N$ and

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L \implies \lim_{n \to \infty} b_n = L$$

<u>Theorem</u>: If $\lim_{n \to \infty} |a_n| = 0 \Rightarrow \lim_{n \to \infty} a_n = 0$

<u>Find Limits</u>: $a_n = \frac{n}{n+1}$, $a_n = \frac{\ln(n)}{n}$, $a_n = (-1)^n$, $a_n = \frac{(-1)^n}{n}$

<u>Theorem</u>: If $\lim_{n \to \infty} a_n = L$ and f is a continuous function at L

then $\lim_{n \to \infty} f(a_n) = f(\lim_{n \to \infty} a_n) = f(L)$

Ex) <u>Find Limit</u> $a_n = \frac{n!}{n^n}$ (Squeeze Thm $\frac{1}{n}\left(\frac{2 \cdot 3 \cdot 4 \cdots n}{n \cdot n \cdot n \cdots n}\right)$)

$a_n = \sin(\pi/n)$

<u>Geometric sequence</u>: The sequence $\{r^n\}_{n=1}^{\infty}$, r = constant is a geometric sequence.

It is convergent for $|r| < 1$ and $r = 1$

<u>or</u> $-1 < r \leq 1$

A sequence is <u>increasing</u> if $a_n < a_{n+1}$ for all $n \geq 1$

A sequence is <u>decreasing</u> if $a_{n+1} < a_n$ for all $n \geq 1$

A sequence is <u>monotonic</u> if it is either increasing or decreasing.

Ex) Show $\left\{\frac{3}{n+5}\right\}$ and $\left\{\frac{n}{n^2+1}\right\}$ are decreasing.

a) NTS: $\frac{3}{n+5} > \frac{3}{(n+1)+5} \Rightarrow \frac{3}{n+5} > \frac{3}{(n+1)+5} = \frac{3}{n+6}$

or $n+5 < n+6 \Rightarrow \frac{1}{n+6} < \frac{1}{n+5} \Rightarrow \frac{3}{(n+1)+5} < \frac{3}{n+5}$ ✓

b) <u>Method 1</u>: $\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$ <u>WTS</u>

$\Leftrightarrow (n+1)(n^2+1) < n(n+1)^2+n$

$\Leftrightarrow n^3+n^2+n+1 < n^3+2n^2+2n$

$\Leftrightarrow 1 < n^2+n$ true for all $n \geq 1 \Rightarrow a_{n+1} < a_n$

<u>Method 2</u>: $f(x) = \frac{x}{x^2+1} \Rightarrow f'(x) = \frac{1-x^2}{(x^2+1)^2} < 0$ for $x^2 > 1$

$\Rightarrow f(x)$ decreasing on $(1, \infty)$

$\Rightarrow f(n) > f(n+1) \Rightarrow \{a_n\}$ decreasing.

A sequence is <u>bounded above</u> if $\exists M \in \mathbb{R}$ s.t. $a_n \leq M$ for all $n \geq 1$

is <u>bounded below</u> if $\exists m \in \mathbb{R}$ s.t. $a_n \geq m$ for all $n \geq 1$

<u>Theorem</u>: Every bounded monotone sequence is convergent.

Ex) Let $\{a_n\}$ be defined as $a_1 = 2$
$a_{n+1} = \frac{1}{2}(a_n + 6)$ $n = 1, 2, 3, \ldots$

determine convergence or divergence.

<u>Solution:</u> Prove by induction. Show increasing

① Base Case: $a_2 = \frac{1}{2}(a_1 + 6) = \frac{1}{2}(2+6) = 4 > a_1 = 2$ ✓

② Assumption: $n = k$ is true ie) $a_{k+1} > a_k$

③ Prove for $n = k+1$

$$a_{k+1} > a_k$$
$$a_{k+1} + 6 > a_k + 6$$
$$\frac{1}{2}(a_{k+1} + 6) > \frac{1}{2}(a_k + 6)$$
$$a_{k+2} > a_{k+1}$$ ✓ $\Rightarrow$ increasing by induction.

Show $\{a_n\}$ is <u>bounded</u>. We will show $a_n < 6$ for all n
(Lower bound is 2 since a_n increasing)

Again, by induction:

① Obviously $a_1 < 6 \Rightarrow a_1 = 2 < 6$

② Assume true for $n = k \Rightarrow$ $a_k < 6$

③ Prove for $n = k+1$

$$a_k < 6 \Rightarrow a_k + 6 < 12$$
$$\frac{1}{2}(a_k + 6) < \frac{1}{2}12$$
$$a_{k+1} < 6$$ ✓

So by Thm, seq is convergent.

<u>BUT</u>, we do not know its Limit!! Thm does <u>not</u> establish this.

Call the limit L, ie. $\lim_{n\to\infty} a_n = L$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n + 6) = \frac{1}{2}\left(\left(\lim_{n\to\infty} a_n\right) + 6\right)$$
$$= \frac{1}{2}(L + 6)$$

Since $a_n \to L$ as $n \to \infty$
$a_{n+1} \to L$ as well

$$\Rightarrow L = \frac{1}{2}(L+6)$$
$$\frac{1}{2}L = 3 \Rightarrow L = 6$$ ✓