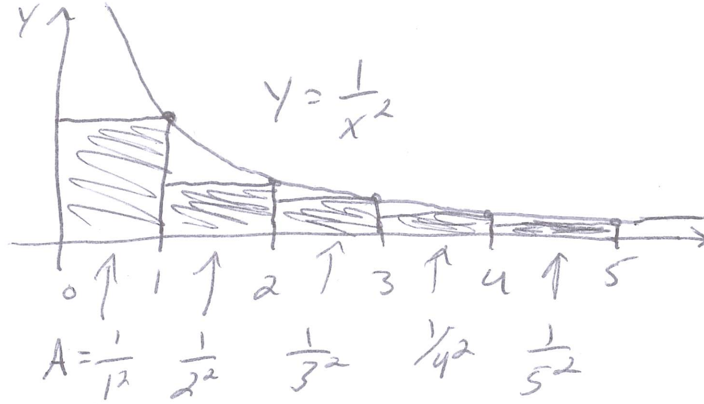
Section 8.3 - Integral and Comparison Tests

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$



$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} + \dots$$

is the area of each box for a Riemann sum added up.

The area of the rectangles is less than $\int_1^{\infty} \frac{1}{x^2}\,dx$ for $x \geq 1$

ie: $\sum_{n=1}^{\infty} \frac{1}{n^2} \leq \int_1^{\infty} \frac{1}{x^2}\,dx$, so if we know the integral, we can determine convergence.

Integral Test: Suppose f is continuous, positive, and decreasing on $[1, \infty)$ and $a_n = f(n)$

then,
(i) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent
(ii) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent

Examples:
1. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$
2. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\quad p > 1$, $p \leq 1$
3. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$
4. $\sum_{n=1}^{\infty} n e^{-n^2}$

Comparison Tests

Comparison Test: Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

(i) If $\sum b_n$ converges and $a_n \leq b_n$ for all n, then $\sum a_n$ converges

(ii) If $\sum b_n$ diverges and $a_n \geq b_n$ for all n, then $\sum a_n$ diverges

Comparisons:
- Geometric $|r|<1$ converge otherwise diverge
- p-series $p>1$, $p\leq 1$

Examples: $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$, $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$, $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

Cannot compare for $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ inequality is wrong way

Limit Comparison Test: Suppose $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where c is a finite number and $c > 0$

then both series converge or both series diverge

Ex) $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

$\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{n^5+5}}$

$\sum_{n=1}^{\infty} \frac{\ln(n)}{n^{3/2}}$ (note: $\ln(n) < n^c$ for $c > 0$)
$n^{1/4}$ in numerator for ex.