Section 8.4 - Ratio and Root Tests

Ratio Test: ① If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ (absolutely) converges

② If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges

③ If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio test is inconclusive
i.e we can't say anything.

Examples: $\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{3^n}$ $\qquad$ $\sum_{n=1}^{\infty}\frac{4^n n!n!}{(2n)!}$ $\qquad$ $\left(\frac{a_{n+1}}{a_n}=\frac{2n+2}{2n+1} \Rightarrow \text{increasing } a_n \geq a_1 = 2 \text{ Test for divergence}\right)$

$\sum_{n=1}^{\infty}\frac{n^n}{n!}$ $\qquad$ $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2 2^n}{n!}$

Root Test: ① If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ is (absolutely) convergent

② If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ is divergent

③ If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1 \Rightarrow$ inconclusive

Note: If either the Ratio Test or Root test is $L=1$, do not try the other test, as it will also give $L=1$.

Examples: $\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n \Rightarrow \sqrt[n]{|a_n|} = \frac{2n+3}{3n+2}$

$\sum_{n=1}^{\infty}\frac{2^n}{n^3}$ $\qquad$ $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$