MATH 009C **PRACTICE FINAL** **August 12, 2017**

Name: ____________________________ **Score:** _______ / 100

Student ID: ____________________________

DO NOT OPEN THE EXAM UNTIL YOU ARE TOLD TO DO SO

	1	2	3	4	5	6	7	8	9	Total
✓										200
Score										
Pts. Possible	25	25	25	25	25	25	25	25	25	210

INSTRUCTIONS FOR STUDENTS

- Questions are on both sides of the paper. This is an 9 question exam.
- Students have 2 hours to complete the exam.
- The test will be out of **200** points (8 questions). You may attempt a 9^{th} question, which will have a maximum of 10 possible points. The highest possible score is therefore **210** points.
- In the above table, the row with the ✓, is for you to keep track of the problems you are attempting/completing.
- You may complete parts of problems, as partial credit will be given based on correctness, completeness, and ideas that are leading to the correct solutions.
- **PLEASE SHOW ALL WORK**. Any unjustified claims will receive no credit. This means you need to state which test you are using for series questions! Clearly box your final answer.
- No notes, textbooks, phones, calculators, etc. are allowed for the exam.
- The back of the test can be used for scratch work.

GOOD LUCK!

FORMULAS:

Common Taylor Series	Common Taylor Series
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, for all $\lvert x\rvert < 1$	$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, for all $x \in \mathbb{R}$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, for all $x \in \mathbb{R}$	$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, for all $x \in \mathbb{R}$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$, for $x \in (-1, 1]$	$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$, for $\lvert x\rvert \le 1$
$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, for $\lvert x-a\rvert < R$	$(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n$, for $\lvert x\rvert < 1$

1) (10 pts.) (a) Determine whether the sequence converges or diverges:

$$a_n = \frac{(2n-1)!}{(2n+1)!}.$$

(15 pts.) (b) Determine whether the sequence converges or diverges:

$$a_n = \left(1 + \frac{2}{n}\right)^n$$

Solution:

(a) By using factorial properties we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(2n-1)!}{(2n+1)!} = \lim_{n\to\infty} \frac{(2n-1)!}{(2n+1)(2n)(2n-1)!} = \lim_{n\to\infty} \frac{1}{2n(2n+1)} = 0$$

So the sequence a_n converges to 0.

(b) Use the exponential-logarithm trick for limits:

$$\lim_{x\to\infty} e^{\ln\left(\left(1+\frac{2}{x}\right)^x\right)} = \lim_{x\to\infty} e^{x\ln\left(1+\frac{2}{x}\right)}$$
$$= e^{\lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{1/x}}$$
$$= e^{\lim_{x\to\infty} \frac{2}{1+2/x}} = e^2$$

where we have applied L'Hopital's rule once. So the sequence converges to e^2.

2) (10 pts.) Determine whether the series is convergent or divergent:

$$\sum_{n=1}^{\infty} \sqrt[n]{2}$$

(15 pts.) (b) Determine whether the series is convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{1+6^n}{7^n}.$$

Solution:

(a) Use the Test for Divergence:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt[n]{2} = \lim_{n\to\infty} 2^{1/n} = 2^0 = 1 \neq 0.$$

Since the limit is not equal to zero, the series diverges.

(b) The series can be divided into 2 geometric series:

$$\sum_{n=1}^{\infty} \frac{1+6^n}{7^n} = \sum_{n=1}^{\infty} \frac{1}{7^n} + \frac{6^n}{7^n} = \sum_{n=1}^{\infty} \left(\frac{1}{7}\right)^n + \sum_{n=1}^{\infty} \left(\frac{6}{7}\right)^n,$$

and since both geometric series converge because $1/7 < 1$ and $6/7 < 1$, so the original series is convergent.

3) (25 pts.) Determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{1}{n^2+4}.$$

Solution:

Use the Integral Test. Compare with $f(x) = \dfrac{1}{x^2+4}$.

- $f(x)$ is continuous on $(-\infty, \infty)$ as it is never undefined.
- $f(x)$ is positive as the denominator is positive for any x.
- $f'(x) = \dfrac{-2x}{(x^2+4)^2} < 0$ for $x > 0 \quad \Rightarrow \quad f(x)$ is decreasing.

Then by computation

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^2+4}\, dx &= \lim_{t\to\infty} \frac{1}{2} \arctan\left(\frac{1}{2}x\right) \Big|_1^t \\
&= \frac{1}{2} \lim_{t\to\infty} \arctan\left(\frac{1}{2}t\right) - \arctan\left(\frac{1}{2}\right) \\
&= \frac{\pi}{2} - \arctan\left(\frac{1}{2}\right).
\end{aligned}$$

So the series is convergent by Integral Test.

4) (25 pts.) Determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}}$$

Solution:

Use the Limit Comparison Test with $b_n = \dfrac{n}{\sqrt[3]{n^7}} = \dfrac{n}{n^{7/3}} = \dfrac{1}{n^{4/3}}$:

$$\begin{aligned}
\lim_{n\to\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}} \cdot \frac{n^{4/3}}{1} &= \lim_{n\to\infty} \frac{n^{7/3}+5n^{4/3}}{\sqrt[3]{n^7+n^2}} \\
&= \lim_{n\to\infty} \frac{n^{7/3}+5n^{4/3}}{\sqrt[3]{n^7+n^2}} \cdot \frac{\frac{1}{n^{7/3}}}{\frac{1}{n^{7/3}}} \\
&= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{\sqrt{1+\frac{1}{n^5}}} \\
&= 1 < \infty
\end{aligned}$$

Therefore, since the limit is positive and finite, and the series $\sum \dfrac{1}{n^{4/3}}$ is a convergent p-series, the original series converges by Limit Comparison Test.

5) (15 pts.) (a) Determine whether the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=1}^{\infty} \frac{2^n n!}{(n+2)!}$$

(10 pts.) (b) Determine whether the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$$

Solution:

(a) Use the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{2^{n+1}(n+1)!}{(n+3)} \cdot \frac{(n+2)!}{2^n n!} \\
&= \lim_{n\to\infty} \frac{(n+2)!(n+1)!}{(n+3)!n!} \cdot \frac{2^{n+1}}{2^n} \\
&= \lim_{n\to\infty} 2\frac{n+1}{n+3} = 2 > 1.
\end{aligned}$$

So the series is divergent by the Ratio Test.

(b) Use the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{(n!)^n}{n^{4n}} \right)^{1/n} = \lim_{n\to\infty} \frac{n!}{n^4} = \lim_{n\to\infty} \frac{n(n-1)(n-2)(n-3)(n-4)\dots}{n \cdot n \cdot n \cdot n} = \infty$$

So the series diverges by the Root Test.

6) (25 pts.) Determine whether the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=2}^{\infty}(-1)^n \frac{n^n}{n!}$$

Solution:

Check absolute convergence: Take absolute value:

$$\sum_{n=2}^{\infty}\left|(-1)^n \frac{n^n}{n!}\right| = \sum_{n=2}^{\infty}\frac{n^n}{n!}.$$

Now do comparison

$$\frac{n^n}{n!} = \frac{n \cdot n \cdot n \cdot \ldots \cdot n}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n} \geq n \quad \Rightarrow \quad \lim_{n\to\infty} n = \infty < \lim_{n\to\infty}\frac{n^n}{n!}$$

Which implies that the series is divergent. Therefore, the series *does not* converge absolutely. Now apply the Test for Divergence to the original series,

$$\lim_{n\to\infty}\frac{n^n}{n!} = \infty$$

by using the $\frac{a_{n+1}}{a_n}$ trick (see the Notes!). Therefore, the original series diverges by Test for Divergence.

7) (25 pts.) Find the radius of convergence and interval of convergence for the following power series:

$$\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$$

Solution:

Use the Ratio Test:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| (-1)^{n+1} \frac{(x-3)^{n+1}}{2(n+1)+1} \cdot \frac{2n+1}{(-1)^n (x-3)^n} \right| \\ &= \lim_{n\to\infty} |x-3| \cdot \frac{2n+1}{2n+3} \\ &= |x-3| \lim_{n\to\infty} \frac{2n+1}{2n+3} = |x-3|. \end{aligned}$$

From the Ratio Test, if the limit is less than 1, the series converges, so we have $|x-3| < 1$, so $R = 1$. Solving the inequality, we have that the *tentative* interval of convergence is $2 < x < 4$. Now we check the endpoints.

$$x = 2 \quad \Rightarrow \quad \sum_{n=0}^{\infty} (-1)^n \frac{(-1)^n}{2n+1} = \sum_{n=0}^{\infty} \frac{1}{2n+1} \quad \Rightarrow \quad \text{divergent by Comparison Test } \sum 1/n$$

$$x = 4 \quad \Rightarrow \quad \sum_{n=0}^{\infty} (-1)^n \frac{1^n}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \quad \Rightarrow \quad \text{cond. convergent by Alt. Series Test}$$

Therefore, the interval convergence is $(2, 4]$.

8) Find the sum of the following series:

(5 pts.) (a) $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$

(5 pts.) (b) $\sum_{n=1}^{\infty}\frac{(-1)^n\sqrt{3}^{2n+1}}{2n+1}$

(10 pts.) (c) $\sum_{n=1}^{\infty}(-1^n)x^{2n}$

Solution:

(a) Use Formula 1, Column 1 from Table:

$$\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{2}\right)^{n-1} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

Or use the geometric series formula $\sum ar^n = \dfrac{a}{1-r}$

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2.$$

(b) Use Formula 3, Column 2 from Table:

$$\sum_{n=1}^{\infty}\frac{(-1)^n\sqrt{3}^{2n+1}}{2n+1} = \arctan(\sqrt{3}) = \frac{\pi}{3}$$

(c) Use Formula 3, Column 2 from Table:

$$\begin{aligned}
\arctan(x) &= \sum_{n=1}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} \\
\frac{d}{dx}\arctan(x) &= \frac{d}{dx}\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} \\
&= \sum_{n=1}^{\infty}(-1)^n x^{2n} \\
\frac{1}{1+x^2} &= \sum_{n=1}^{\infty}(-1)^n x^{2n}
\end{aligned}$$

Or do a substitution of $-x^2$ into Formula 1, Column 1 to get the same result.

9) (20 pts.) (a) Compute the following integral using Taylor series.

$$\int \arctan(x^2)\ dx$$

(5 pts.) (b) Find the Taylor series centered at $a = \frac{\pi}{2}$ for

$$f(x) = \sin(x)$$

Solution:

(a) We know from the table on the front

$$\arctan(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\arctan(x^2) = \sum_{n=0}^{\infty}(-1)^n \frac{(x^2)^{2n+1}}{2n+1}$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{2n+1}$$

$$\int \arctan(x^2)\ dx = \int \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{2n+1}\ dx$$

$$= \sum_{n=0}^{\infty}(-1)^n \int \frac{x^{4n+2}}{2n+1}\ dx$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)}$$

(b) We know from the table on the front

$$\sin(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

By using the derivatives and the definition and computing $f^{(n)}(a)$, all of the even terms stay and the odds are zero, so we get

$$\sin(x) = \sum_{n=1}^{\infty}(-1)^n \frac{(x-\frac{\pi}{2})^{2n}}{(2n)!}$$

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END OF TEST