MATH 009C **PRACTICE MIDTERM** **July 13, 2017**

Name: ____________________________ **Score:** _______ / 100

Student ID: ____________________________

DO NOT OPEN THE EXAM UNTIL YOU ARE TOLD TO DO SO

	1	2	3	4	5	Total
✓						
Score						
Pts. Possible	25	25	25	25	25	110

INSTRUCTIONS FOR STUDENTS

- You can use both sides of the paper for your solution. This is an **4** question exam.
- Students have **50** minutes to complete the exam.
- The test will be out of **100** points (4 questions). You may attempt a 5^{th} question, which will have a maximum of 10 possible points. The highest possible score is therefore **110** points.
- In the above table, the row with the ✓, is for you to keep track of the problems you are attempting/completing.
- You may complete parts of problems, as partial credit will be given based on correctness, completeness, and ideas that are leading to the correct solutions.
- **PLEASE SHOW ALL WORK**. Any unjustified claims will receive no credit.
- No notes, textbooks, phones, calculators, etc. are allowed for the exam.
- The last page of the test can be used for scratch work.

GOOD LUCK!

1) (25 pts.) Determine if the integral is convergent or divergent. If it converges, compute the integral:

$$\int_1^\infty \frac{x}{\sqrt{x^6+1}}\,dx$$

Solution:

Use the Direct Comparison Test (or Limit Comparison Test) for Integrals.

$$x^6 < x^6+1 \;\Rightarrow\; \sqrt{x^6} < \sqrt{x^6+1} \;\Rightarrow\; \frac{1}{\sqrt{x^6+1}} < \frac{1}{\sqrt{x^6}} \;\Rightarrow\; \frac{x}{\sqrt{x^6+1}} < \frac{x}{\sqrt{x^6}} = \frac{1}{x^2} \quad \text{for all } x > 1.$$

So then by computation,

$$0 < \int_1^\infty \frac{x}{\sqrt{x^6+1}}\,dx < \int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} -\frac{1}{t} + 1 = 1 < \infty.$$

So by Direct Comparison Test, the integral converges.

2) (20 pts.) (a) Eliminate the parameter in the for the following parametric equation

$$x = 9\cos(t) + 4 \quad y = 9\sin(t) + 1$$

(5 pts.) (b) Identify the type of graph from your result in part (a).

Solution:

(a) Rewrite x and y in terms of $\cos^2(t)$ and $\sin^2(t)$:

$$\begin{aligned} x &= 9\cos(t) + 4 \\ (x-4)^2 &= 81\cos^2(t) \\ \frac{(x-4)^2}{81} &= \cos^2(t) \end{aligned}$$

and

$$\begin{aligned} y &= 9\sin(t) + 1 \\ (y-1)^2 &= 81\sin^2(t) \\ \frac{(y-1)^2}{81} &= \sin^2(t) \end{aligned}$$

By using the trig identity $\sin^2(t) + \cos^2(t) = 1$:

$$\frac{(x-4)^2}{81} + \frac{(y-1)^2}{81} = 1$$

$$(x-4)^2 + (y-1)^2 = 81$$

(b) The graph is an circle centered at $(4, 1)$ with radius 9.

3) Consider the parametric defined as: $x = 4 + t^2, y = t^2 + t^3$

(10 pts.) (a) Compute $\dfrac{dy}{dx}$.

(10 pts.) (b) Compute $\dfrac{d^2y}{dx^2}$.

(5 pts.) (c) For which values of t is the curve concave upward?

Solution:

(a) Use the formula for the derivative:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t + 3t^2}{2t} = 1 + \frac{3}{2}t$$

(b) Use the formula for the second derivative:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(1 + \frac{3}{2}t)}{2t} = \frac{3}{4t}$$

(c) We need to find where $\dfrac{d^2y}{dx^2} > 0$. This is straightforward; the curve is concave upward for $t > 0$.

4) (10 pts.) (a) Consider the polar curve $r = 2\sin(\theta)$. Find the slope of the tangent line to the curve at the point $\theta = \dfrac{\pi}{6}$.

(10 pts.) (b) Consider the polar curve $r = 1 + \cos(\theta)$. Find the values of θ where the tangent line is vertical for $0 \le \theta < 2\pi$.

Solution:

(a) Use the formula for the derivative:

$$\begin{aligned}
\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)} &= \frac{2\cos(\theta)\sin(\theta) + 2\sin(\theta)\cos(\theta)}{2\cos^2(\theta) - 2\sin^2(\theta)} \\
&= \frac{4\cos(\theta)\sin(\theta)}{2(1 - 2\sin^2(\theta))} \\
&\overset{\theta=\pi/6}{=} \frac{4 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2}}{2(1 - 2 \cdot \frac{1}{4})} = \sqrt{3}
\end{aligned}$$

Alternatively, since $x = r\cos(\theta) = 2\sin(\theta)\cos(\theta) = \sin(2\theta)$ and $y = r\sin(\theta) = 2\sin^2(\theta)$:

$$\begin{aligned}
\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2 \cdot (2\sin(\theta)\cos(\theta))}{2\cos(2\theta)} = \frac{\sin(2\theta)}{\cos(2\theta)} &= \tan(2\theta) \\
\overset{\theta=\pi/6}{=} \tan(\pi/3) &= \sqrt{3}
\end{aligned}$$

(b) Use the formula for the derivative:

$$\begin{aligned}
\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)} &= \frac{-\sin(\theta)\sin(\theta) + (1 + \cos(\theta))\cos(\theta)}{-\sin(\theta)\cos(\theta) - (1 + \cos(\theta))\sin(\theta)} \\
&= \frac{-\sin^2(\theta) + \cos(\theta) + \cos^2(\theta)}{-\sin(\theta)\cos(\theta) - \sin(\theta) - \sin(\theta)\cos(\theta)} \\
&= \frac{2\cos^2(\theta) + \cos(\theta) - 1}{-\sin(\theta)(2\cos(\theta) + 1)} \\
&= \frac{(2\cos(\theta) - 1)(\cos(\theta) + 1)}{-\sin(\theta)(2\cos(\theta) + 1)}
\end{aligned}$$

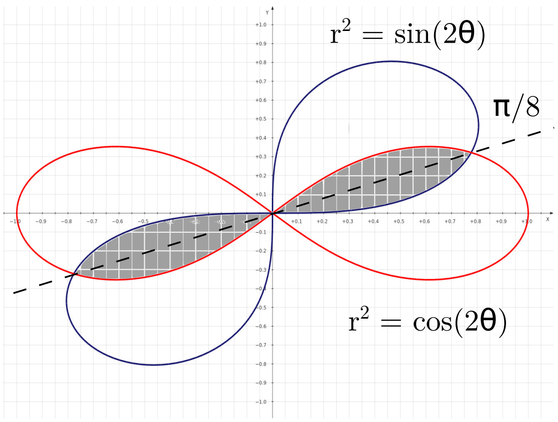
We only need to find where the denominator is zereo, we only solve two equations:

$$\begin{aligned}
2\cos(\theta) + 1 &= 0 & -\sin(\theta) &= 0 \\
\cos(\theta) &= -\frac{1}{2} & \sin(\theta) &= 0 \\
\theta &= \frac{2\pi}{3}, \frac{4\pi}{3} & \theta &= 0, \pi
\end{aligned}$$

5) (25 pts.) Find the area of the region that lies inside the petals (the region is shaded in the labeled plot below).

Hint: Use the symmetry at $\frac{\pi}{8}$ *to simplify the integral. How many pieces are there?*

$$r^2 = \cos(2\theta)$$
$$r^2 = \sin(2\theta)$$



Solution:

Find the intersection point first (the picture already gives you the value). We have to solve the equation:

$$\begin{aligned}\sin(2\theta) &= \cos(2\theta)\\ \tan(2\theta) &= 1\\ 2\theta &= \frac{\pi}{4}\\ \theta &= \frac{\pi}{8}\end{aligned}$$

Using hint 1, we have 2 petals, but by cleverly choosing the second function, which starts at the origin at $\theta = 0$, from $\theta = 0$ to $\theta = \pi/8$, we get half of the first petal. Since the polar rose is symmetric, really have 4 half petals. Now we compute

$$\begin{aligned}A &= 2 \cdot 2 \int_0^{\pi/8} \frac{1}{2} r^2 \, d\theta\\ &= 4 \int_0^{\pi/8} \frac{1}{2} \sin(2\theta) \, d\theta\\ &= 2 \int_0^{\pi/8} \sin(2\theta) \, d\theta\\ &= -\cos(2\theta)|_0^{\pi/8}\\ &= 1 - \frac{\sqrt{2}}{2}\end{aligned}$$

THIS PAGE IS LEFT BLANK FOR ANY SCRATCH WORK

END OF TEST