LAST NAME: FIRST NAME:

MATH 009C - Summer 2017

Quiz 2: July 6, 2017

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1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. For what values of t is the curve concave up?

$$x = 2\sin(t), \quad y = 3\cos(t), \quad 0 < t < 2\pi$$

Solution: By direct calculation

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-3\sin(t)}{2\cos(t)} = -\frac{3}{2}\tan(t)$$
$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{-\frac{3}{2}\sec^2(t)}{2\cos(t)} = -\frac{3}{4}\sec^3(t)$$

The curve is concave up where

$$-\sec^3(t) > 0 \Rightarrow \sec^3(t) < 0 \Rightarrow \sec(t) < 0 \Rightarrow \frac{1}{\cos(t)} < 0,$$

which is for $\frac{\pi}{2} < t < \frac{3\pi}{2}$.

2. Compute the length of the curve defined by the following parametric equations:

$$x = e^t\cos(t) \qquad y = e^t\sin(t) \quad \text{for } 0 \leq t \leq 2\pi$$

Solution: By direct computation

$$\begin{aligned}
L &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{(e^t\cos(t) - e^t\sin(t))^2 + (e^t\sin(t) + e^t\cos(t))^2}\, dt \\
&= \int_0^{2\pi} \sqrt{e^{2t}(\cos(t) - \sin(t))^2 + e^{2t}(\sin(t) + \cos(t))^2}\, dt \\
&= \int_0^{2\pi} e^t\sqrt{\cos^2(t) + \sin^2(t) - 2\sin(t)\cos(t) + \sin^2(t) + \cos^2(t) + 2\sin(t)\cos(t)}\, dt \\
&= \int_0^{2\pi} e^t\sqrt{2(\cos^2(t) + \sin^2(t))}\, dt \\
&= \sqrt{2}\int_0^{2\pi} e^t\, dt \\
&= \sqrt{2}(e^{2\pi} - 1)
\end{aligned}$$

Please, show all work.

3. Find the surface area of the solid you get by rotating the following parametric curve around the x-axis for $-2 \le t \le 0$:

$$x = 4t^2 - 1 \qquad y = 3 - 2t$$

Solution:

By using the formula

$$\begin{aligned}
2\pi \int_a^b g(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt &= 2\pi \int_{-2}^{0} (3-2t)\sqrt{(8t)^2 + (-2)^2}\, dt \\
&= 2\pi \int_{-2}^{0} (3-2t)\sqrt{64t^2+4}\, dt \\
&= 4\pi \int_{-2}^{0} (3-2t)\sqrt{16t^2+1}\, dt \\
&= 12\pi \int_{-2}^{0} \sqrt{16t^2+1}\, dt - 8\pi \int_{-2}^{0} t\sqrt{16t^2+1}\, dt \\
&= 3\pi \int_{\arctan(-8)}^{0} \sec^3(\theta)\, d\theta - \frac{\pi}{4}\int_{65}^{1} \sqrt{u}\, du \\
&= \frac{3\pi}{2}\left(\sec(\theta)\tan(\theta) - \ln[\sec(\theta)+\tan(\theta)]\right)\Big|_{\arctan(-8)}^{0} \\
&\quad - \left(\frac{\pi}{6}u^{\frac{3}{2}}\right)\Big|_{65}^{1} \\
&= 12\pi\sqrt{65} + \frac{3\pi}{2}\ln(8+\sqrt{65}) - \frac{\pi}{6}(65\sqrt{65}-1)
\end{aligned}$$

4. Find the slope of the tangent line to the given polar curve at the specified angle θ:

$$r = 2\sin(\theta) \qquad \theta = \frac{\pi}{6}$$

Solution: By direct computation:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)} \\
&= \frac{2\cos(\theta)\sin(\theta) + 2\sin(\theta)\cos(\theta)}{2\cos^2(\theta) - 2\sin^2(\theta)} \\
&= \frac{4\cos(\theta)\sin(\theta)}{2(\cos^2(\theta) - \sin^2(\theta))} \\
&= \frac{2\sin(2\theta)}{2(\cos^2(\theta) - \sin^2(\theta))} \\
&= \frac{2\frac{\sqrt{3}}{2}}{2\left(\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right)} \\
&= \sqrt{3}
\end{aligned}$$