LAST NAME: FIRST NAME:

MATH 009C - Summer 2017

Quiz 4: July 18, 2017

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1. Determine whether the sequence converges or diverges. If it converges, find its limit.

$$(a) \qquad a_n = n^2 e^{-n}$$

$$(b) \qquad a_n = n \sin\left(\frac{1}{n}\right)$$

$$(c) \qquad a_n = \frac{n!}{2^n}$$

Solution: Use direct computation of limits for (a) and (b). For (c), use inequalities.

(a) By direct computation and L'Hopital's Rule

$$\begin{aligned}\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 e^{-n} &= \lim_{x\to\infty} \frac{x^2}{e^x} \\ &= \lim_{x\to\infty} \frac{1}{e^x} = 0 \qquad \text{convergent}\end{aligned}$$

(b) By direct computation and substitution $u = \dfrac{1}{n}$, we have

$$\begin{aligned}\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) &= \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \\ &= \lim_{u\to 0} \frac{\sin(u)}{u} = 1 \qquad \text{convergent}\end{aligned}$$

(c) $a_n = \dfrac{n!}{2^n} = \dfrac{1}{2} \cdot \dfrac{2}{2} \cdot \dfrac{3}{2} \cdot \dfrac{4}{2} \cdot \ldots \cdot \dfrac{(n-1)}{2} \cdot \dfrac{n}{2} \geq \dfrac{1}{2} \cdot \dfrac{n}{2} = \dfrac{n}{4} \to \infty \quad \text{as} \quad n \to \infty$

Where the inequality follows from the fact that the terms from $n = 2$ to the $(n-1)^{\text{st}}$ term multiplied together must be greater than or equal to 1/2, since the first is 1/2, and the subsequent fractions are all bigger than 1. Therefore the sequence is greater than $n/4$, which is divergent, so then a_n is divergent.

Please, show all work.

2. Find the general term a_n of the sequence:

$$(a) \qquad \{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots\}$$

$$(b) \qquad \{2, 7, 12, 17\}$$

$$(c) \qquad \{1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \ldots\}$$

Solution: We will assume all series start with $n = 1$, and the element a_1.

(a) The numerator is always 1, and the denominator is all the positive odd numbers starting at one. Odd numbers are usually represented by $2n + 1$, but since the first value of denominator is 1 for $n = 1$, we have the denominator is $2n - 1$. So,

$$a_n = \frac{1}{2n - 1}$$

(b) There is only a numerator. We can see that the difference between each element is 5. This can be described as a recursive sequence

$$a_1 = 2 \qquad a_n = a_{n-1} + 5 \qquad \text{for } \ n \geq 2$$

The series can also be written in terms of an arithmetic sequence

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= 2 + 5(n-1) \\ &= 5n - 3 \end{aligned}$$

(c) This sequence is geometric, and alters signs. Since the sequence alternates $+, -, +, -, \ldots$ we will have an $(-1)^{n+1}$ term. The sequence in the numerator is $1, 2, 4, 8, \ldots$, which is powers of 2, hence 2^{n-1}. The bottom is powers of 3, so we have

$$\begin{aligned} a_n &= (-1)^{n+1} \frac{2^{n-1}}{3^{n-1}} \\ &= (-1)^{n+1} \left(\frac{2}{3}\right)^{n-1} \end{aligned}$$