LAST NAME: FIRST NAME:

MATH 009C - Summer 2017

Quiz 5: July 25, 2017

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1. Determine whether the series converges or diverges. If it converges, find its sum. (Show all the steps!)

$$\sum_{n=1}^{\infty}\left(\frac{1}{e^n}+\frac{1}{n(n+1)}\right)$$

Solution: First note the the first term is a geometric series, and the second is a telescoping series. We can only break the series into two parts if both of the series are convergent. So we deal with them separately. First the geometric series

$$\sum_{n=1}^{\infty}\frac{1}{e^n}=\sum_{n=1}^{\infty}\left(\frac{1}{e}\right)^n=\sum_{n=1}^{\infty}\frac{1}{e}\left(\frac{1}{e}\right)^{n-1}=\frac{1/e}{1-1/e}=\frac{1}{e-1}\quad \text{convergent since } \frac{1}{e}<1$$

The second series requires that we use the definition of a series, a limit of partial sums. The second series has the general term $a_n=\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$, then the partial sum s_n is

$$\begin{aligned} s_n &= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots+\left(\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &= 1-\frac{1}{n+1}\end{aligned}$$

Then by taking the limit of partial sums, we have

$$\lim_{n\to\infty}1-\frac{1}{n+1}=1$$

So since both series are convergent, we can break the sum, and we have already computed the sums individually, hence

$$\sum_{n=1}^{\infty}\left(\frac{1}{e^n}+\frac{1}{n(n+1)}\right)=\sum_{n=1}^{\infty}\frac{1}{e^n}+\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\frac{1}{e-1}+1=\frac{e}{e-1}$$

Please, show all work.

2. Determine whether the series converges or diverges.

$$(a) \qquad \sum_{n=1}^{\infty} ne^{-n}$$

$$(b) \qquad \sum_{n=1}^{\infty} \frac{\arctan(n)}{n^3}$$

$$(c) \qquad \sum_{n=1}^{\infty} \frac{1+5^n}{1+2^n}$$

Solution:

(a) Use the integral test with $f(x) = xe^{-x}$. The function $f(x)$ needs to be positive for $x > 1$. Exponentials are always positive, and $x > 0$ since we are only considering $x > 1$. The function $f(x)$ is continuous since it is the product of continuous functions. Lastly, the derivative is decreasing

$$f'(x) = e^{-x} + (-1)xe^{-x} = e^{-x}(1-x) < 0 \qquad \text{for } x > 1.$$

Therefore, we apply the integral test and integrate the improper integral

$$\begin{aligned}
\int_1^{\infty} xe^{-x} &= \lim_{t\to\infty} \int_1^t xe^{-x}\,dx \\
&= \lim_{t\to\infty} -e^{-x}(x+1)\Big|_1^t \qquad \text{(integration by parts)} \\
&= -\lim_{t\to\infty} e^{-t}(t+1) + \frac{2}{3} \\
&= 0 + \frac{2}{e} = \frac{2}{e} \qquad \text{convergent}
\end{aligned}$$

(b) Apply the Comparison Test with $b_n = \dfrac{1}{n^3}$, which is a convergent p-series since $p > 1$. Recall, that arctangent is bounded above, ie. $\arctan(x) < \frac{\pi}{2}$, therefore, we have that

$$\sum_{n=1}^{\infty} \frac{\arctan(x)}{n^3} \leq \sum_{n=1}^{\infty} \frac{\pi/2}{n^3} = \frac{\pi}{2}\sum_{n=1}^{\infty} \frac{1}{n^3} < \infty$$

Therefore, by Comparison Test, the original series is convergent.

(c) Compute the limit

$$\begin{aligned}
\lim_{n\to\infty} \frac{1+5^n}{1+2^n} &= \lim_{n\to\infty} \frac{1+5^n}{1+2^n} \cdot \frac{\frac{1}{5^n}}{\frac{1}{5^n}} \\
&= \frac{0+1}{0+0} = \infty
\end{aligned}$$

So by Test for Divergence, the series is divergent.

Please, show all work.