

## MATH 146B

## HW#1 Solutions

Section 4.1 # 7, 11, 12

Section 4.2 # 11, 14, 16, 22, 24

Section 4.1

# 7) Determine whether the given functions are linearly dependent or independent. If linearly dependent, find a relation among them.

Solution: Compute the Wronskian for  $f_1, f_2, f_3$  given as  
 $f_1 = 2t - 3, f_2 = t^2 + 1, f_3 = 2t^2 - t$

$$\begin{aligned} W(f_1, f_2, f_3) &= \begin{vmatrix} 2t-3 & t^2+1 & 2t^2-t \\ 2 & 2t & 4t-1 \\ 0 & 2 & 4 \end{vmatrix} \\ &= 0 \begin{vmatrix} t^2+1 & 2t^2-t \\ 2t & 4t-1 \end{vmatrix} - 2 \begin{vmatrix} 2t-3 & 2t^2-t \\ 2 & 4t-1 \end{vmatrix} + 4 \begin{vmatrix} 2t-3 & t^2+1 \\ 2 & 2t \end{vmatrix} \\ &= -2(4t^2 - 12t + 3) + 4(2t^2 - 6t - 2) = -14 \neq 0 \end{aligned}$$

$\Rightarrow$  linearly independent  $\blacksquare$

11) Verify that the given functions are solutions to the differential equation and determine their Wronskian

$$y''' + y' = 0 ; \quad 1, \cos(t), \sin(t)$$

Solution:

(2)

$$\text{For } y=1 \Rightarrow \frac{d^3}{dt^3}(1) + \frac{d}{dt}(1) = ? \quad 0 + 0 = 0 \quad 0 = 0 \checkmark$$

$$\text{For } y=\cos(t) \Rightarrow \frac{d^3}{dt^3}(\cos(t)) + \frac{d}{dt}(\cos(t)) = ? \quad \sin(t) + (-\sin(t)) = 0 \quad 0 = 0 \checkmark$$

$$\text{For } y=\sin(t) \Rightarrow \frac{d^3}{dt^3}(\sin(t)) + \frac{d}{dt}(\sin(t)) = ? \quad -\cos(t) + \cos(t) = 0 \quad 0 = 0 \checkmark$$

All are solutions  
to the ODE.

$$W(f_1, f_2, f_3) = \begin{vmatrix} 1 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \\ 0 & -\cos(t) & -\sin(t) \end{vmatrix} = 1 \begin{vmatrix} -\sin(t) & \cos(t) \\ -\cos(t) & -\sin(t) \end{vmatrix} \\ = \sin^2(t) + \cos^2(t) \\ = 1 \neq 0 \Rightarrow \text{linearly independent.}$$

12) Verify that the given functions are solutions to the ODE and determine their Wronskian.  $y^{(u)} + y'' = 0$

$\begin{matrix} 1 \\ \cos(t) \\ \sin(t) \end{matrix}$

$$\text{Solution: For } y=1 \Rightarrow (1)^{(u)} + \frac{d^2}{dt^2}(1) = 0 \stackrel{?}{=} 0 \checkmark$$

All functions work

$$\text{For } y=t \Rightarrow (t)^{(u)} + \frac{d^2}{dt^2}(t) = 0 \stackrel{?}{=} 0 \checkmark$$

$$\text{For } y=\sin(t) \Rightarrow (\sin(t))^{(u)} + \frac{d^2}{dt^2}\sin(t) = \sin(t) - \sin(t) = 0 \stackrel{?}{=} 0 \checkmark$$

$$\text{For } y=\cos(t) \Rightarrow (\cos(t))^{(u)} + \frac{d}{dt}\cos(t) = \cos(t) - \cos(t) = 0 \stackrel{?}{=} 0 \checkmark$$

$$W(f_1, f_2, f_3, f_4) = \begin{vmatrix} 1 & t & \cos(t) & \sin(t) \\ 0 & 1 & -\sin(t) & \cos(t) \\ 0 & 0 & -\cos(t) & -\sin(t) \\ 0 & 0 & \sin(t) & -\cos(t) \end{vmatrix} = 1 \neq 0 \Rightarrow \text{linearly independent.}$$

(3)

Section 4.2 - For all of these problems, find the general solutions of the given ODE (3)

$$11) \quad y''' - y'' - y' + y = 0$$

Solution: Use the characteristic equation

$$\begin{aligned} \Rightarrow r^3 - r^2 - r + 1 &= 0 \\ -r^2(r+1) + (-r+1) &= 0 \\ (-r+1)(-r^2+1) &= 0 \\ \Rightarrow r=1, r=\pm 1 \end{aligned} \quad \left| \begin{array}{l} \Rightarrow y(t) = C_1 e^t + C_2 t e^t + C_3 e^{-t} \\ \text{is the general solution since} \\ r=1 \text{ is a double root (multiplicity 2)} \\ \text{and only one copy of } r=-1. \end{array} \right.$$

$$14) \quad y^{(4)} - 4y''' + 4y'' = 0$$

Solution: Use characteristic equation or reduction of order.

By reduction of order, let  $u = y'' \Rightarrow u' = y''', u'' = y^{(4)}$

$\Rightarrow$  ODE becomes  $u'' - 4u' + 4u = 0 \Rightarrow u = \tilde{C}_1 e^{2t} + \tilde{C}_2 t e^{2t}$

$\Rightarrow y'' = \tilde{C}_1 e^{2t} + \tilde{C}_2 t e^{2t} \xrightarrow{\text{integrate}} y' = \frac{1}{2} \tilde{C}_1 e^{2t} + \tilde{C}_2 \left( -\frac{1}{4} e^{2t} + \frac{1}{2} t e^{2t} \right) + C_2$

$\Rightarrow y' = \left( \frac{1}{2} \tilde{C}_1 - \frac{1}{4} \tilde{C}_2 \right) e^{2t} + \frac{1}{2} \tilde{C}_2 t e^{2t} + C_2$

$\xrightarrow{\text{integrate}} y = \frac{1}{2} \left( \frac{1}{2} \tilde{C}_1 - \frac{1}{4} \tilde{C}_2 \right) e^{2t} + \frac{1}{2} \tilde{C}_2 \left( -\frac{1}{4} e^{2t} + \frac{1}{2} t e^{2t} \right) + C_2 t + C_1$

$\Rightarrow y = \underbrace{\frac{1}{2} \left( \frac{1}{2} \tilde{C}_1 - \frac{1}{4} \tilde{C}_2 \right)}_{\text{constant}} e^{2t} + \underbrace{\frac{1}{4} \tilde{C}_2 t e^{2t}}_{\text{constant}} + C_2 t + C_1$

$\Rightarrow y(t) = C_1 + C_2 t + C_3 e^{2t} + C_4 t e^{2t}$

(4)

$$16) y^{(4)} - 5y'' + 4y = 0$$

Solution: Use characteristic equation.

$$\Rightarrow r^4 - 5r^2 + 4 = 0 \quad \text{Let } s = r^2 \Rightarrow s^2 = r^4$$

$$\Rightarrow s^2 - 5s + 4 = 0$$

$$(s-1)(s-4) = 0$$

$$s=1, s=4 \Rightarrow r^2=1 \text{ and } r^2=4 \Rightarrow r=\pm 1, r=\pm 2$$

$$\Rightarrow \boxed{y(t) = c_1 e^{-t} + c_2 e^t + c_3 e^{-2t} + c_4 e^{2t}}$$

$$22) y^{(4)} + 2y'' + y = 0$$

Solution: Use characteristic equation

$$\Rightarrow r^4 + 2r^2 + 1 = 0$$

$$\Rightarrow (r^2 + 1)^2 = 0$$

$$\Rightarrow (r^2 + 1)(r^2 + 1) = 0$$

$$r^2 + 1 = 0$$

$$\Rightarrow r = \pm i \text{ (multiplicity 2)}$$

$$\Rightarrow \boxed{y(t) = (c_1 + c_2 t) \cos(t) + (c_3 + c_4 t) \sin(t)}$$

is the general solution since  
 $r = \pm i$ , but we have both of  
these roots twice.

$$24) y''' + 5y'' + 6y' + 2y = 0$$

Solution: Use characteristic equation.

$$r^3 + 5r^2 + 6r + 2 = 0 \quad \text{By rational roots theorem rational roots can be}$$

$$\Rightarrow (r+1)(r^2 + 4r + 2) = 0$$

$$\Rightarrow r = -1 \quad r = -2 \pm \sqrt{2}$$

$$\Rightarrow \boxed{y(t) = c_1 e^{-t} + c_2 e^{(-2+\sqrt{2})t} + c_3 e^{(-2-\sqrt{2})t}}$$

$$\frac{P}{q} = \pm \left( \frac{2, 1}{1} \right) = \pm 2, \pm 1$$

$$\text{Note: } r = -1 \Rightarrow (-1)^3 + 5(-1)^2 + 6(-1) + 2$$

$$= -1 + 5 - 6 + 2 = 0$$

$\Rightarrow -1$  is a root