

MATH 146B

HW #2 Solutions

Section 4.3 # 2, 8, 17

Section 4.4 # 6, 14

Section 4.3

#2) Determine the general solution of: $y^{(4)} - y = 3t + \cos(t)$

Solution: Using undetermined coefficients:

Find the homogeneous solution by solving

$$y_H^{(4)} - y_H = 0 \Rightarrow r^4 - 1 = 0 \Rightarrow (r^2 - 1)(r^2 + 1) = 0 \Rightarrow r = \pm 1, r = \pm i$$

$$\Rightarrow y_H(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos(t) + c_4 \sin(t)$$

For particular solution, note there is duplication of $\cos(t)$ in the ~~general~~ ^{homogeneous} solution and the RHS of the ODE. So, choose

$$y_p(t) = A + Bt + t(C \cos(t) + D \sin(t))$$

$$\Rightarrow y_p^{(4)} = (4C + Dt) \sin(t) + (Ct - 4D) \cos(t)$$

$$\Rightarrow y_p^{(4)} - y = -A - Bt - 4D \cos(t) + 4C \sin(t)$$

$$\Rightarrow \left. \begin{array}{l} -A = 0 \\ -B = 3 \end{array} \right\} \begin{array}{l} -4D = 1 \\ 4C = 0 \end{array} \quad \left. \begin{array}{l} \\ \end{array} \right\} \begin{array}{l} \text{Comparing to RHS} \\ \text{of ODE} \end{array}$$

$$\Rightarrow A = 0, B = -3, C = 0, D = -\frac{1}{4}$$

$$\Rightarrow y_G(t) = y_H(t) + y_p(t)$$

$$\Rightarrow y_G(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos(t) + c_4 \sin(t) - 3t - \frac{1}{4} t \sin(t)$$

8) Find the general solution of: $y^{(4)} + y''' = \sin(2t)$

Solution: Using undetermined coefficients:

Find homogeneous solution by solving

$$y_H^{(4)} + y_H''' = 0 \Rightarrow r^4 + r^3 = 0 \Rightarrow r^3(r+1) = 0 \Rightarrow r = 0, 0, 0, -1$$

$$\Rightarrow y_H(t) = C_1 + C_2 t + C_3 t^2 + C_4 e^{-t}$$

For particular solution, note there is no duplication of terms in $y_H(t)$ and RHS of ODE, so choose

$$y_p(t) = A \cos(2t) + B \sin(2t) \Rightarrow \begin{aligned} y_p^{(4)}(t) &= 16A \cos(2t) + 16B \sin(2t) \\ y_p'''(t) &= 8A \sin(2t) - 8B \cos(2t) \end{aligned}$$

$$\Rightarrow y_p^{(4)} + y_p''' = (16A - 8B) \cos(2t) + (16B + 8A) \sin(2t)$$

$$\Rightarrow \begin{aligned} 16A - 8B &= 0 \\ 16B + 8A &= 1 \end{aligned} \Rightarrow \begin{aligned} A &= \frac{1}{40} \\ B &= \frac{1}{20} \end{aligned}$$

$$\Rightarrow y_G(t) = y_H(t) + y_p(t) = C_1 + C_2 t + C_3 t^2 + C_4 e^{-t} + \frac{1}{20} \sin(2t) + \frac{1}{40} \cos(2t)$$

17) Determine a suitable form of $Y(t)$ if the method of undetermined coefficients is to be used:

$$y^{(4)} - y''' - y'' + y' = t^2 + 4 + t \sin(t)$$

Solution: Find the homogeneous solution:

$$r^4 - r^3 - r^2 + r = 0 \Rightarrow r(r^3 - r^2 - r + 1) = 0 \Rightarrow r(r-1)(r^2-1) = 0$$

$$\Rightarrow r = 0, 1, 1, -1$$

$$\Rightarrow y_H(t) = C_1 + (C_2 + C_3 t) e^t + C_4 e^{-t}$$

For the particular solution, note there is duplication in the polynomial terms, but not with trig functions.

Therefore, we would choose

$$Y_p(t) = Y(t) = t(A + Bt + Ct^2) + (D + Et)\cos(t) + (F + Gt)\sin(t)$$

Section 4.4

6) Use variation of parameters to find the general solution of $y^{(4)} + 2y'' + y = \sin(t)$.

Solution: We know the homogeneous solution $Y_H(t)$ is

$$Y_H(t) = (c_1 + c_2 t)\cos(t) + (c_3 + c_4 t)\sin(t)$$

from HW #1, Section 4.2 #22.

To use variation of parameters, we can follow Ex 1, p. 241.

We seek a particular solution of the form

$$Y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + u_3(t)y_3(t) + u_4(t)y_4(t)$$

where the $u_i(t)$ are unknowns and $y_i(t)$ are the fundamental set of solutions. So for us,

$$y_1(t) = c_1 \cos(t) \quad y_2(t) = c_3 \sin(t) \\ y_3(t) = c_2 t \cos(t) \quad y_4(t) = c_4 t \sin(t)$$

$$\text{Each } u_i(t) \text{ is given by } u_i(t) = \int \frac{g(t) W_i(t)}{W(t)} dt \quad (*)$$

where $g(t) = \sin(t)$ (RHS of ODE)

$W(t)$ = Wronskian of fundamental solution set

$W_i(t)$ = Wronskian of fund. solution set with i^{th} column replaced by $(0, 0, 0, 1)^T$

$$W(t) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ \cos(t) & \sin(t) & t \cos(t) & t \sin(t) \\ -\sin(t) & \cos(t) & \cos(t) & \sin(t) + t \cos(t) \\ -\cos(t) & -\sin(t) & -2 \sin(t) & 2 \cos(t) - t \sin(t) \\ \sin(t) & -\cos(t) & -3 \cos(t) & -3 \sin(t) - t \cos(t) \end{vmatrix} = 4$$

(4)

Using Mathematica, and the sign matrix $\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$

$$W_1(t) = -2 \sin(t) + 2t \cos(t)$$

$$W_2(t) = 2 \cos(t) + 2t \sin(t)$$

$$W_3(t) = -2 \cos(t)$$

$$W_4(t) = -2 \sin(t)$$

Use formula

$$\int \frac{g(t) W_i(t)}{W(t)} dt = u_i(t), \quad g(t) = \sin(t)$$

to get

$$u_1(t) = \frac{1}{8} [3 \sin(t) \cos(t) - 2t \cos^2(t) - t]$$

$$u_2(t) = \frac{1}{8} [\sin^2(t) - 2 \cos^2(t) - 2t \sin(t) \cos(t) + t^2]$$

$$u_3(t) = -\frac{\sin^2(t)}{4}, \quad u_4(t) = \frac{1}{4} [\cos(t) \sin(t) - t]$$

$$Y_p(t) = \sum_{i=1}^4 Y_i(t) u_i(t) = \frac{1}{8} [\sin(t) - 3t \cos(t) - t^2 \sin(t)]$$

Since the $\sin(t)$ and $3t \cos(t)$ term show up in $Y_H(t)$

$$\Rightarrow Y_p(t) = -\frac{1}{8} t^2 \sin(t)$$

$$\Rightarrow Y_G(t) = Y_H(t) + Y_p(t) = (c_1 + c_2 t) \cos(t) + (c_3 + c_4 t) \sin(t) - \frac{1}{8} t^2 \sin(t)$$

□

14) Find a formula involving integrals for a particular solution of $y''' - y'' + y' - y = g(t)$

(5)

Solution: Find the ~~general~~ homogeneous solution $y_H(t)$

$$\Rightarrow r^3 - r^2 + r - 1 = 0 \Rightarrow (r-1)(r^2+1) = 0 \Rightarrow r = 1, \pm i$$

$$\Rightarrow y_H(t) = C_1 e^t + C_2 \cos(t) + C_3 \sin(t) \Rightarrow \begin{aligned} y_1(t) &= e^t \\ y_2(t) &= \cos(t) \\ y_3(t) &= \sin(t) \end{aligned}$$

$$\text{So } y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t) + u_3(t) y_3(t)$$

$$\text{where } u_i(t) = \int_{t_0}^t \frac{g(s) W_i(s)}{W(s)} ds$$

$$W(s) = 2e^s, W_1(s) = 1, W_2(s) = -(e^s \cos(s) - e^s \sin(s))$$

$$W_3(s) = -(e^s \cos(s) + e^s \sin(s))$$

$$\Rightarrow y_p(t) = e^t \int_{t_0}^t \frac{g(s)(1)}{2e^s} ds + \cos(t) \int_{t_0}^t \frac{g(s)(-(e^s \cos(s) - \sin(s)))}{2e^s} ds + \sin(t) \int_{t_0}^t \frac{g(s)(-(e^s \cos(s) + e^s \sin(s)))}{2e^s} ds$$

$$\Rightarrow y_p(t) = \frac{1}{2} \int_{t_0}^t [e^{t-s} - \sin(t-s) - \cos(t-s)] g(s) ds$$

where we have rearranged pieces and use identities

$$\sin(\alpha \pm \beta) = \sin(\alpha) \cos(\beta) \pm \cos(\alpha) \sin(\beta)$$

$$\cos(\alpha \pm \beta) = \cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta)$$