

MATH 146B

HW #3 Solutions

Section 5.1 # 2, 3, 7, 25, 26

Section 5.2 # 25, 26

Section 5.3 # 2, 6, 8

Section 5.1

Find the radius of convergence for each power series.

2) The series is $\sum_{n=0}^{\infty} \frac{n}{2^n} x^n$. Use Ratio Test.

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{n+1}{2^{n+1}} x^{n+1}}{\frac{n}{2^n} x^n} \right| = \lim_{n \rightarrow \infty} \frac{n+1}{n} \cdot \frac{2^n}{2^{n+1}} \cdot \left| \frac{x^{n+1}}{x^n} \right|$$

$$= \frac{|x|}{2} \lim_{n \rightarrow \infty} \frac{n+1}{n} = \frac{|x|}{2} < 1 \quad \begin{matrix} \text{by Ratio Test} \\ \text{(to have convergence)} \end{matrix}$$

$$\Rightarrow |x| < 2 \Rightarrow R = 2$$

3) The series is $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$. Use Ratio Test.

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{2(n+1)}}{(n+1)!}}{\frac{x^{2n}}{n!}} \right| = \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} \cdot \frac{|x|^{2n+2}}{|x|^{2n}}$$

$$= \lim_{n \rightarrow \infty} \frac{|x|^2}{n+1} = 0 < 1 \quad \text{for all } x \quad \begin{matrix} \text{by Ratio Test} \\ \Rightarrow R = \infty \end{matrix}$$

7) The series is $\sum_{n=1}^{\infty} \frac{(-1)^n n^2 (x+2)^n}{3^n}$. Use Ratio Test.

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+1} (n+1)^2 (x+2)^{n+1}}{3^{n+1}}}{\frac{(-1)^n n^2 (x+2)^n}{3^n}} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)^2}{n^2} \cdot \frac{3^n}{3^{n+1}} \cdot \frac{|x+2|^{n+1}}{|x+2|^n}$$

$$= \frac{|x+2|}{3} \lim_{n \rightarrow \infty} \frac{(n+1)^2}{n^2} = \frac{|x+2|}{3} < 1 \quad \begin{matrix} \text{by Ratio Test} \\ \Rightarrow |x+2| < 3 \Rightarrow R = 3 \end{matrix}$$

(2)

Rewrite the given expression as a sum whose generic term involves x^n .

$$25) \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x \sum_{K=1}^{\infty} K a_K x^{K-1}$$

I think the indices should all be n , not m and K , to get solution, so

$$\begin{aligned} & \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} \\ = & \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n \\ = & \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n \\ = & \boxed{\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + n a_n] x^n} \end{aligned}$$

* add 2 inside series 1, and subtract 2 on summation to re-index
* multiply x into second series

* First series same.
Second series constant at $n=0$
since the $n=0$ term is just 0

* Combine series, since both start at $n=0$,
and have x^n factor.

$$26) \sum_{n=1}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n$$

* Multiply x through second series

* Reindex series 1 by adding 1 inside, subtracting 1 on series.

* Reindex series 2 by subtracting 1 inside, adding 1 on series.

$$\begin{aligned} & \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\ = & \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n \\ = & a_1 + \sum_{n=1}^{\infty} (n+1)a_{n+1} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n \\ = & \boxed{a_1 + \sum_{n=1}^{\infty} [(n+1)a_{n+1} + a_{n-1}] x^n} \end{aligned}$$

* Write out $n=0$ term for series 1

* Combine series and factor.

Section 5.2 -

25) Plot several partial sums in a series solution of the given IVP.
about the point $x=0$.

$$y'' + xy' + 2y = 0, \quad y(0)=0, \quad y'(0)=1$$

Solution: We will routinely use the following substitutions

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \quad \textcircled{A}$$

Plugging \textcircled{A} into the ODE

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n = 0$$

$$\Rightarrow 2(a_2 + a_0) + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} + (n+2)a_n] x^n = 0$$

$$\Rightarrow 2(a_2 + a_0) = 0 \quad \text{and} \quad (n+2)(n+1) a_{n+2} + (n+2)a_n = 0$$

$$\Rightarrow a_2 = -a_0 \quad \text{and} \quad a_{n+2} = \frac{-a_n}{n+1}$$

$$\Rightarrow a_3 = -\frac{a_1}{2}, \quad a_4 = \frac{-a_2}{3 \cdot 1} = \frac{a_0}{3 \cdot 1}, \quad a_5 = \frac{-a_3}{4} = \frac{a_1}{4 \cdot 2}$$

~~$$a_6 = \frac{-a_4}{5} = \frac{-a_0}{5 \cdot 3 \cdot 1}, \quad a_7 = \frac{-a_5}{6} = \frac{-a_1}{6 \cdot 4 \cdot 2}$$~~

$$\Rightarrow y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \dots$$

$$\Rightarrow y(x) = a_0 + a_1 x + (-a_0)x^2 + \left(-\frac{a_1}{2}\right)x^3 + \left(\frac{a_0}{3 \cdot 1}\right)x^4 + \left(\frac{a_1}{4 \cdot 2}\right)x^5 + \dots$$

$$y(0) = 0 \Rightarrow a_0 = 0, \quad y'(0) = 1 \Rightarrow a_1 = 1$$

$$\Rightarrow y(x) = x - \frac{1}{2}x^3 + \frac{1}{4 \cdot 2}x^5 - \frac{1}{6 \cdot 4 \cdot 2}x^7 + \dots$$

see last page for plots.



(4)

26) Plot several partial sums in a series solution of the given IVP about the point $x=0$.

$$(4-x^2)y'' + 2y = 0, y(0)=0, y'(0)=1$$

Solution: Plug in \star from #25 into this ODE

$$\begin{aligned} & \Rightarrow (4-x^2) \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2 \sum_{n=0}^{\infty} a_n x^n = 0 \\ & \Rightarrow \sum_{n=2}^{\infty} 4n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0 \\ & \Rightarrow \sum_{n=2}^{\infty} 4(n+2)(n+1)a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0 \\ & \Rightarrow (8a_2 + 2a_0) + (24a_3 + 2a_1)x + \sum_{n=2}^{\infty} \left[4(n+2)(n+1)a_{n+2} - (n(n-1)-2)a_n \right] x^n = 0 \\ & \quad 24a_3 + 2a_1 = 0 \quad a_{n+2} = \frac{(n(n-1)-2)a_n}{4(n+2)(n+1)} = \frac{(n^2-n-2)a_n}{4(n+2)(n+1)} \\ & \Rightarrow 8a_2 + 2a_0 = 0 \quad a_3 = -\frac{a_1}{12} \quad = \frac{(n+2)(n+1)a_n}{4(n+2)(n+1)} = \frac{(n-2)}{4(n+2)} a_n \\ & \Rightarrow a_2 = -\frac{a_0}{4} \end{aligned}$$

$$\begin{aligned} y(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots \\ &= a_0 + a_1 x + \left(-\frac{a_0}{4}\right) x^2 + \left(\frac{-a_1}{12}\right) x^3 + 0 \cdot x^4 + \left(-\frac{a_1}{12 \cdot 20}\right) x^5 + \dots \end{aligned}$$

$$\Rightarrow y_1(x) = a_0 + \left(-\frac{a_0}{4}\right) x^2$$

$$y_2(x) = a_1 x - \frac{a_1}{12} x^3 - \frac{a_1}{12 \cdot 20} x^5 - \dots$$

$$y(0) = 0 \Rightarrow a_0 = 0 \Rightarrow y_1(x) = 0$$

$$y'(0) = 1 \Rightarrow a_1 = 1 \Rightarrow y_2(x) = x - \frac{1}{12} x^3 - \frac{1}{240} x^5 - \frac{1}{2040} x^7 \dots$$

~~see plots~~ see last pages for plots.

Section 5.3

For question 2, it will be helpful to read p. 265, the bottom half of the page. Work the same method up to Egn 3 on that page.

Note: given $P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0$

$$\Rightarrow \phi''(x) = -\frac{Q(x)}{P(x)}\phi'(x) - \frac{R(x)}{P(x)}\phi(x)$$

2) Determine $\phi''(x_0)$, $\phi'''(x_0)$, $\phi^{(4)}(x_0)$ for the given x_0 if $y = \phi(x)$ is a solution to the given IVP.

$$y'' + \sin(x)y' + \cos(x)y = 0 \quad y(0) = 0$$

$$P(x) = 1, Q(x) = \sin(x), R(x) = \cos(x)$$

$$\Rightarrow \phi''(x) = -\sin(x)\phi'(x) - \cos(x)\phi(x)$$

$$\phi'''(x) = -2\cos(x)\phi'(x) - \sin(x)\phi''(x) + \sin(x)\phi(x)$$

$$\phi^{(4)}(x) = \cos(x)\phi(x) + \sin(x)\phi'(x) + 2\sin(x)\phi'(x) - 2\cos(x)\phi''(x)$$

$$= \cos(x)\phi(x) + 3\sin(x)\phi'(x) - \sin(x)\phi'''(x)$$

$$= \cos(x)\phi(x) + 3\sin(x)\phi'(x) - 3\cos(x)\phi''(x) - \sin(x)\phi'''(x)$$

$$= \cos(x)\phi(x) + 3\sin(x)\phi'(x) - 3\cos(x)\phi''(x) - \sin(x)\phi'''(x)$$

$$\text{So, } y(0) = 0 \Rightarrow \phi(0) = 0 \text{ and } y'(0) = 1 \Rightarrow \phi'(0) = 1$$

$$\therefore \phi''(0) = -\sin(0)\phi'(0) - \cos(0)\phi(0) = 0$$

$$\therefore \phi'''(0) = -2\cos(0)\phi'(0) - \sin(0)\phi''(0) + \sin(0)\phi(0) = -2$$

$$\therefore \phi^{(4)}(0) = \cos(0)\phi(0) + 3\sin(0)\phi'(0) - 3\cos(0)\phi''(0) - \sin(0)\phi'''(0) = 0$$

$$\Rightarrow \boxed{\begin{aligned} \phi''(0) &= 0 \\ \phi'''(0) &= -2 \\ \phi^{(4)}(0) &= 0 \end{aligned}}$$

(6)

- 6) Determine a lower bound for the radius of convergence of the series solution about the given x_0 for the ODE
- $$(x^2 - 2x - 3)y'' + xy' + 4y = 0 \quad x_0 = 4, x_0 = -4, x_0 = 0$$

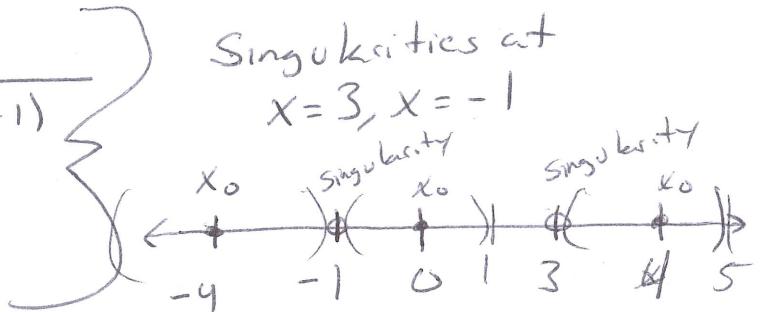
Solution: From standard form,

$$\frac{Q(x)}{P(x)} = \frac{x}{x^2 - 2x - 3} = \frac{x}{(x-3)(x+1)}$$

$$\frac{R(x)}{P(x)} = \frac{4}{(x-3)(x+1)}$$

Singularities at

$$x = 3, x = -1$$



So for $x_0 = 4$, appeal to the diagram to the right. We can only expand from $x_0 = 4$ up to 3 before hitting a singularity. For $x_0 = 0$, we can also only expand until we reach $x = -1$, a singularity, and a distance of 1. For $x_0 = -4$, we expand to $x_0 = -1$, which is a distance of 3 units. So for

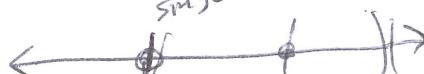
$$x_0 = 4 \Rightarrow R = 1 \text{ is a bound}$$

$$x_0 = 0 \Rightarrow R = 1 \text{ is a bound}$$

$$x_0 = -4 \Rightarrow R = 3 \text{ is a bound. } \square$$

- 8) Same question as 6, but for $xy'' + y = 0, x_0 = 1$

Solution: $\frac{Q(x)}{P(x)} = \frac{0}{x} = 0 \text{ (if } x \neq 0\text{)} \quad \frac{R(x)}{P(x)} = \frac{1}{x} \quad \left. \begin{array}{l} \text{Singularity} \\ \text{at } x = 0 \end{array} \right\}$



The biggest neighborhood we can draw about $x_0 = 1$, is $x < 1$, thus radius 1 which will reach the singularity at $x = 0$, thus

$$R = 1 \text{ is a bound. } \square$$

(7)

Section 5.4

Determine the general solution of the ODE that is valid in any interval not including the singular point.

$$3) x^2 y'' - 3xy' + 4y = 0$$

Solution: Assume a solution of the form $y = x^r$

$$\Rightarrow y' = rx^{r-1}, y'' = r(r-1)x^{r-2}$$

$$\Rightarrow x^2(r(r-1)x^{r-2}) - 3x(rx^{r-1}) + 4x^r = 0$$

$$\Rightarrow [r(r-1) - 3r + 4]x^r = 0 \Rightarrow [r^2 - 4r + 4]x^r = 0$$

$$\Rightarrow (r-2)^2 x^r = 0 \Rightarrow r=2, 2 \text{ repeated roots}$$

$$\text{general form is } y = (C_1 + C_2 \ln|x|)x^r, \text{ so } y(x) = (C_1 + C_2 \ln|x|)x^2$$

$$11) x^2 y'' + 2xy' + 4y = 0$$

Solution: Assume a solution of the form $y = x^r$

$$\Rightarrow y' = rx^{r-1}, y'' = r(r-1)x^{r-2}$$

$$\Rightarrow x^2(r(r-1)x^{r-2}) + 2x(rx^{r-1}) + 4x^r = 0$$

$$\Rightarrow [r(r-1) + 2r + 4]x^r = 0 \Rightarrow [r^2 + r + 4]x^r = 0$$

$$\Rightarrow r^2 + r + 4 = 0 \Rightarrow r = -\frac{1}{2}(1 \pm i\sqrt{15}) \text{ complex roots.}$$

$$\text{general form is } y = C_1 x^{\lambda} \cos(\mu \ln|x|) + C_2 x^{\lambda} \sin(\mu \ln|x|) \quad x > 0$$

$r = \lambda \pm i\mu$

$$\Rightarrow y(x) = C_1 |x|^{-\frac{1}{2}} \cos\left(\frac{1}{2}\sqrt{15} \ln|x|\right) + C_2 |x|^{-\frac{1}{2}} \sin\left(\frac{\sqrt{15}}{2} \ln|x|\right)$$

25) Find all singular points of the ODE and determine whether each one is regular or irregular.

$$(x+2)^2(x-1)y'' + 3(x-1)y' - 2(x+2)y = 0$$

Solution: Look at ratios

$$p(x) = \frac{Q(x)}{P(x)} = \frac{3(x-1)}{(x+2)^2(x-1)} = \frac{3}{(x+2)^2} \quad g(x) = \frac{R(x)}{P(x)} = \frac{-2(x+2)}{(x+2)^2(x-1)} = \frac{-2}{(x+2)(x-1)}$$

} \Rightarrow Singular points
are at
 $x = -2, 1$

For $x = -2$,

$$\lim_{x \rightarrow -2} (x+2)p(x) = \lim_{x \rightarrow -2} (x+2) \cdot \frac{3}{(x+2)^2} = \infty$$

$\Rightarrow x = -2$ is an irregular singular point.

For $x = 1$

$$\lim_{x \rightarrow 1} (x-1)p(x) = \lim_{x \rightarrow 1} \frac{3(x-1)}{(x+2)^2} = 0$$

$$\lim_{x \rightarrow 1} (x-1)^2 g(x) = \lim_{x \rightarrow 1} \frac{-2(x-1)^2}{(x+2)(x-1)} = \lim_{x \rightarrow 1} \frac{-2(x-1)}{(x+2)} = 0$$

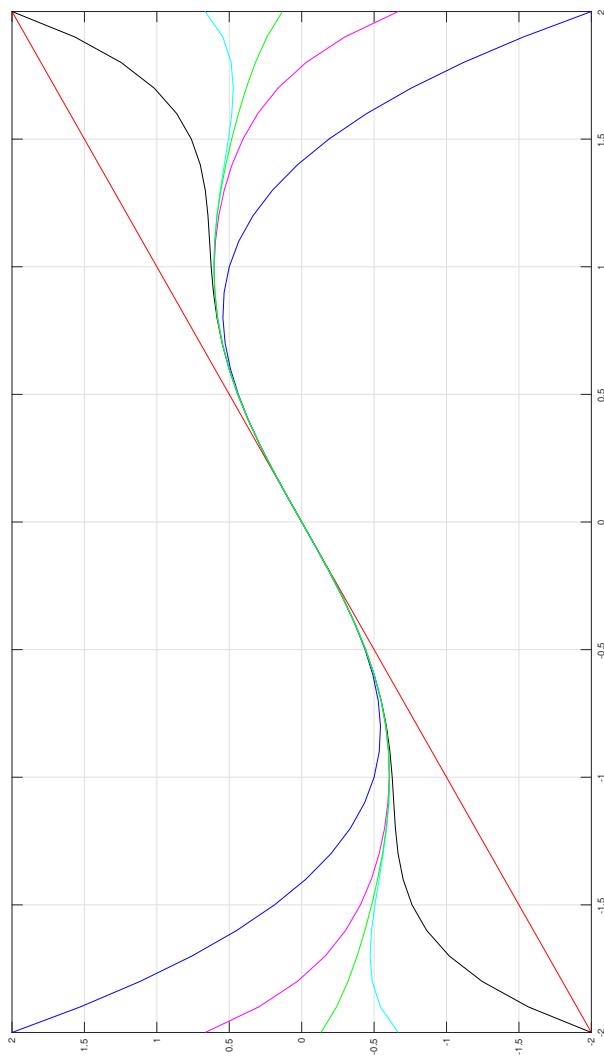
$\Rightarrow x = 1$ is a regular singular point.

MATH 146B - Ordinary and Partial Differential Equations II

Plots For Homework 3

All of the plots are created using the file MATH146B_HW3.m. All plots are inserted into the document in landscape format.

Section 5.2, # 25



Section 5.2, # 26

