

HW #4 Solutions

Section 5.5 #1, 4, 8, 14

Section 10.1 #2, 10, 11, 14, 18

Section 5.5

1) $2xy'' + y' + xy = 0$

Solution: First we check that $x_0 = 0$ is a regular singular point

$$\left. \begin{aligned} \lim_{x \rightarrow 0} x \frac{Q(x)}{P(x)} &= \lim_{x \rightarrow 0} \frac{x}{2x} = \frac{1}{2} < \infty \\ \lim_{x \rightarrow 0} x^2 \frac{R(x)}{P(x)} &= \lim_{x \rightarrow 0} x^2 \frac{x}{2x} = 0 < \infty \end{aligned} \right\} \Rightarrow x_0 = 0 \text{ regular singular point}$$

Now assume a series solution as in the book, $a_0 \neq 0$

$$y = \sum_{n=0}^{\infty} a_n x^{r+n}, \quad y' = \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1}, \quad y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n-2} \quad (\star)$$

Substituting into the ODE,

$$\sum_{n=0}^{\infty} 2(r+n)(r+n-1)a_n x^{r+n-1} + \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n+1} = 0$$

$$\sum_{n=0}^{\infty} [2(r+n)(r+n-1)a_n + (r+n)a_n] x^{r+n-1} + \sum_{n=2}^{\infty} a_{n-2} x^{r+n-1} = 0$$

$$\sum_{n=0}^{\infty} [(r+n)(2r+2n-1)a_n] x^{r+n-1} + \sum_{n=2}^{\infty} a_{n-2} x^{r+n-1} = 0$$

$$\Rightarrow r(2r-1)a_0 x^{r-1} + (r+1)(2r+2-1)a_1 x^r + \sum_{n=2}^{\infty} [(r+n)(2r+2n-1)a_n + a_{n-2}] x^{r+n-1} = 0$$

$\Rightarrow r(2r-1) = 0$ Indicial Equation

$r=0, r=\frac{1}{2}$ Exponents at singularity for $x_0 = 0$

At $n=1 \Rightarrow (r+1)(2r+1)a_1 = 0 \Rightarrow a_1 = 0$ for $r = \frac{1}{2}, r=0$

$$\text{At } n \geq 2 \Rightarrow a_n = \frac{-a_{n-2}}{(r+n)(2r+2n-1)}$$

Recurrence Relation (★)(★)

(2)

Note: Since $a_1 = 0 \Rightarrow a_n = 0$ for odd n from (★)(★)

For $r = \frac{1}{2}$ Let $r = \frac{1}{2}$

$$\Rightarrow a_n = \frac{-a_{n-2}}{n(2n+1)} \Rightarrow a_2 = \frac{-a_0}{2 \cdot (5)}, a_4 = \frac{a_0}{(2)(4)(5)(9)}$$

$\downarrow \quad \downarrow \quad \downarrow$
 $2! \quad 2^2 \quad 4m+1$

$$a_6 = \frac{-a_0}{(2)(4)(6)(5)(9)(13)}, \Rightarrow a_{2m} = \frac{(-1)^m a_0}{2^m (m!) (5 \cdot 9 \cdot 13 \cdot \dots \cdot (4m+1))}$$

for $m = 1, 2, 3, \dots$

$$\Rightarrow y_1(x) = a_0 x^{\frac{1}{2}} \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^m (m!) (5 \cdot 9 \cdot 13 \cdot \dots \cdot (4m+1))} \right] \quad m = 1, 2, 3, \dots$$

For $r = 0$ Since $r_1 - r_2 \notin \mathbb{Z}$, $r_2 = 0$ yields another solution.

$$\Rightarrow a_n = \frac{-a_{n-2}}{n(2n-1)} \Rightarrow a_2 = \frac{-a_0}{3 \cdot 2}, a_4 = \frac{a_0}{2^2 2! (3)(7)}$$

$$a_6 = \frac{-a_0}{2^3 3! (3)(7)(11)}, \Rightarrow a_{2m} = \frac{(-1)^m a_0}{2^m (m!) (3)(7) \dots (4m-1)} \quad m = 1, 2, 3, \dots$$

$$\Rightarrow y_2(x) = a_0 \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{(2^m)(m!)(3)(7) \dots (4m-1)} \right] \quad m = 1, 2, 3, \dots$$

4) $xy'' + y' - y = 0$

$\Rightarrow y'' + \frac{1}{x} y' - \frac{1}{x} y = 0$

$\Rightarrow p_0 = \lim_{x \rightarrow 0} x \cdot \frac{1}{x} = 1 < \infty$
 $q_0 = \lim_{x \rightarrow 0} x^2 \left(-\frac{1}{x}\right) = 0 < \infty \Rightarrow x_0 = 0$ is a regular singular point.

\Rightarrow indicial equation is $r(r-1) + r = 0 \Rightarrow r^2 = 0, r = 0$

\Rightarrow exp. of singular point $x_0 = 0$ are $r = 0 \Rightarrow$ only search for $Y_1(x)$

Using the same substitution as in #1, 3, substituting into the ODE, we have

$\sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-1} + \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1} - \sum_{n=0}^{\infty} a_n x^{r+n} = 0$
 $\parallel \parallel \parallel - \sum_{n=1}^{\infty} a_{n-1} x^{r+n-1} = 0$
 $\Rightarrow a_0 [r(r-1) + r] x^{r-1} + \sum_{n=1}^{\infty} [(r+n)(r+n-1) a_n + (r+n) a_n - a_{n-1}] x^{r+n-1} = 0$

\hookrightarrow recovers indicial equation

$\Rightarrow (r+n)^2 a_n - a_{n-1} = 0 \Rightarrow a_n = \frac{a_{n-1}}{(r+n)^2}$ Recurrence Relation
 $n \geq 1$

Let $r=0$ $a_1 = \frac{a_0}{(1)^2}, a_2 = \frac{a_0}{(2)^2}, a_3 = \frac{a_0}{(3 \cdot 2 \cdot 1)^2}, a_4 = \frac{a_0}{(4 \cdot 3 \cdot 2 \cdot 1)^2}$

$\Rightarrow a_n = \frac{a_0}{(n!)^2}$

$\Rightarrow Y_1(x) = a_0 \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$



$$8) 2x^2 y'' + 3xy' + (2x^2 - 1)y = 0$$

Solution: Compute the limits

$$\left. \begin{aligned} \lim_{x \rightarrow 0} x \frac{Q(x)}{P(x)} &= \lim_{x \rightarrow 0} \frac{3}{2} = \frac{3}{2} < \infty \\ \lim_{x \rightarrow 0} x^2 \frac{R(x)}{P(x)} &= \lim_{x \rightarrow 0} x^2 - 1 = 0 < \infty \end{aligned} \right\} \Rightarrow x_0 = 0 \text{ is a regular}$$

Use substitutions from #1, substituting into the ODE

$$\sum_{n=0}^{\infty} 2(r+n)(r+n-1)a_n x^{r+n} + \sum_{n=0}^{\infty} 3(r+n)a_n x^{r+n} + \sum_{n=0}^{\infty} 2a_n x^{r+n+2} - \sum_{n=0}^{\infty} a_n x^{r+n} = 0$$

$$\sum_{n=0}^{\infty} [2(r+n)(r+n-1)a_n + 3(r+n)a_n - a_n] x^{r+n} + \sum_{n=2}^{\infty} 2a_{n-2} x^{r+n} = 0$$

$$a_0 [2r^2 + r - 1] x^r + a_1 [2r^2 + 5r + 2] x^{r+1} + \sum_{n=2}^{\infty} [2(r+n)(r+n-1)a_n + 3(r+n)a_n - a_n + 2a_{n-2}] x^{r+n} = 0$$

Assuming $a_0 \neq 0$,

$$\Rightarrow (r+1)(2r-1) = 0 \quad \text{Indicial Equation}$$

$$r = -1, r = \frac{1}{2}$$

Exponents at singular point for $x_0 = 0$

$$\text{for } n=1: \Rightarrow (2r+1)(r+2)a_1 = 0 \Rightarrow a_1 = 0 \text{ for } r = -1, \frac{1}{2}$$

$$\text{For } n \geq 2: \Rightarrow a_n = \frac{-2a_{n-2}}{(r+n+1)[2(r+n)-1]} \quad \text{for } n=2, 3, 4, \dots$$

So since $a_1 = 0 \Rightarrow$ odd a_n are all zero

$$\text{For } r = \frac{1}{2} \quad a_n = \frac{-2a_{n-2}}{n(2n+3)} \quad n=2, 3, 4, \dots$$

$$\text{So for } m=1, 2, 3, \dots \quad a_{2m} = \frac{-a_{2m-2}}{m(4m+3)} = \frac{(-1)^m a_0}{(m!)(7 \cdot 5 \cdot \dots \cdot (4m+3))}$$

$$\Rightarrow y_1(x) = a_0 x^{\frac{1}{2}} \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{(m!)(7 \cdot 5 \cdot \dots \cdot (4m+3))} \right]$$

$$r = -1$$

$$a_n = \frac{-2a_{n-2}}{n(2n-3)} \quad n = 2, 3, 4, 5, \dots$$

$$\Rightarrow a_{2m} = \frac{-a_{2m-2}}{m(4m-3)} = \frac{(-1)^m a_0}{(m!)(5 \cdot 9 \cdot \dots \cdot (4m-3))}$$

$$\Rightarrow y_2(x) = a_0 x^{-1} \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{(m!)(5 \cdot 9 \cdot \dots \cdot (4m-3))} \right]$$

$$14) x^2 y'' + xy' + x^2 y = 0$$

$$\Rightarrow y'' + \frac{1}{x} y' + y = 0$$

$$\Rightarrow \left. \begin{aligned} p_0 &= \lim_{x \rightarrow 0} x \cdot \frac{1}{x} = 1 < \infty \\ q_0 &= \lim_{x \rightarrow 0} x^2 \cdot 1 = 0 < \infty \end{aligned} \right\} \Rightarrow x_0 = 0 \text{ is a regular singular pt.}$$

Indicial Equation: $r(r-1) + r = 0 \Rightarrow r^2 = 0 \Rightarrow r = 0$

Using the usual substitution as before, $a_0 \neq 0$,

$$\Rightarrow \sum_{n=0}^{\infty} a_n [(r+n)(r+n-1) + (r+n)] x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} = 0$$

$$\Rightarrow a_0 [r(r-1) + r] x^r + a_1 [(r+1)r + r + 1] x^{r+1} + \sum_{n=2}^{\infty} [a_n (r+n)^2 + a_{n-2}] x^{r+n} = 0$$

$$\Rightarrow a_n = -\frac{a_{n-2}}{(r+n)^2} \quad n \geq 2, \quad \oplus \Rightarrow a_1 = 0 \text{ so } a_n = 0 \text{ for all odd } n$$

Let $r=0$ $\Rightarrow a_n = -\frac{a_{n-2}}{n^2}$ or $a_{2m} = -\frac{a_{2m-2}}{(2m)^2} \quad m=1,2,3,\dots$

$$\Rightarrow a_2 = -\frac{a_0}{2^2}, \quad a_4 = \frac{a_0}{2^4 \cdot 2^2}, \quad a_6 = -\frac{a_0}{2^6 (3 \cdot 2)^2}, \dots, \quad a_m = \frac{(-1)^m a_0}{2^{2m} (m!)^2}$$

$$\Rightarrow y_1(x) = a_0 \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} (m!)^2} \right] \quad x > 0$$

(d) Use the root test

$$\lim_{n \rightarrow \infty} |a_n|^{1/n} = \lim_{n \rightarrow \infty} \left| \frac{x^{2n}}{2^{2n} (n!)^2} \right|^{1/n} = \frac{|x|^2}{2^2} \lim_{n \rightarrow \infty} \frac{1}{(n!)^{2/n}} = \frac{|x|^2}{2^2} \left(\lim_{n \rightarrow \infty} \frac{1}{(n!)^{1/n}} \right)^2$$

$$\begin{aligned} \Rightarrow \lim_{n \rightarrow \infty} (n!)^{1/n} &= \lim_{n \rightarrow \infty} \exp \left(\frac{1}{n} (\ln n!) \right) \\ &= \lim_{n \rightarrow \infty} \exp \left(\frac{1}{n} (\ln 1 + \ln 2 + \ln 3 + \dots + \ln n) \right) \quad \left. \begin{array}{l} \text{exp rules} \\ \end{array} \right\} \\ &\geq \lim_{n \rightarrow \infty} \exp \left(\frac{1}{n} \int_1^n \ln x \, dx \right) \\ &= \lim_{n \rightarrow \infty} \exp \left[\frac{n \ln(n) - n + 1}{n} \right] \end{aligned}$$

Section 10.1

7

$$2) y'' + 2y = 0 \quad y'(0) = 1, y'(\pi) = 0$$

BC1 BC2

Solution: $r^2 + 2 = 0 \Rightarrow r = \pm \sqrt{2}i$

$$\Rightarrow y(x) = c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x)$$

Using BC1 $\Rightarrow y'(x) = -\sqrt{2}c_1 \sin(\sqrt{2}x) + \sqrt{2}c_2 \cos(\sqrt{2}x)$

$$\Rightarrow 1 = \sqrt{2}c_2 \Rightarrow c_2 = \frac{1}{\sqrt{2}}$$

Using BC2 $\Rightarrow 0 = -\sqrt{2}c_1 \sin(\sqrt{2}\pi) + \cos(\sqrt{2}\pi)$

$$\Rightarrow c_1 = \frac{1}{\sqrt{2}} \cot(\sqrt{2}\pi)$$

$$y(x) = \frac{1}{\sqrt{2}} \left(\cot(\sqrt{2}\pi) \cos(\sqrt{2}x) + \sin(\sqrt{2}x) \right)$$

$$10) y'' + 3y = \cos(x) \quad y'(0) = 0, y'(\pi) = 0$$

BC1 BC2

Solution: $y_h(x) = c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)$

$$y_p(x) = \frac{1}{2} \cos(x)$$

$$\Rightarrow y(x) = c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x) + \frac{1}{2} \cos(x)$$

$$\Rightarrow y'(x) = -\sqrt{3}c_1 \sin(\sqrt{3}x) + \sqrt{3}c_2 \cos(\sqrt{3}x) - \frac{1}{2} \sin(x)$$

BC1 $\Rightarrow c_2 = 0$, BC2 $\Rightarrow -\sqrt{3}c_1 \sin(\sqrt{3}\pi) = 0$

$$\Rightarrow c_1 = 0$$

$$\Rightarrow y(x) = \frac{1}{2} \cos(x)$$

11) $x^2 y'' - 2xy' + 2y = 0$, $y(1) = -1$, $y(2) = 1$

Solution: Euler DE. From 5.5, we know the general solution

is $y(x) = c_1 x + c_2 x^2$

BC1 $\Rightarrow -1 = c_1 + c_2$
BC2 $\Rightarrow 1 = 2c_1 + 4c_2$
 $\Rightarrow c_1 = -5/2$
 $\Rightarrow c_2 = 3/2$

$\Rightarrow y(x) = -\frac{5}{2}x + \frac{3}{2}x^2$

14) $y'' + \lambda y = 0$, $y(0) = 0$, $y'(\pi) = 0$

Solution: $r^2 + \lambda = 0 \Rightarrow r = \pm \sqrt{-\lambda}$

Look at 3 cases: $\lambda = 0$, $\lambda > 0$, $\lambda < 0$

$\lambda = 0 \Rightarrow r = 0, 0 \Rightarrow y(x) = c_1 + c_2 x$ and BC's imply $c_1 = 0$
 $c_2 = 0$
 \Rightarrow only trivial solutions $y(x) = 0$

$\lambda > 0 \Rightarrow r = \pm \sqrt{\lambda} i \Rightarrow y(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$

Using BC's, $\Rightarrow c_1 = 0$ and $c_2 \sqrt{\lambda} \cos(\sqrt{\lambda} \pi) = 0$

So for non-trivial solutions, assume $c_2 \neq 0$

$\Rightarrow 0 = \cos(\sqrt{\lambda} \pi) \Rightarrow \sqrt{\lambda} \pi = (n - \frac{1}{2}) \pi \Rightarrow \lambda = (n - \frac{1}{2})^2$, $n \in \mathbb{Z}$

$\Rightarrow y(x) = c_2 \sin((n - \frac{1}{2}) \pi x)$

$\lambda < 0 \Rightarrow r = \pm \sqrt{\lambda} \Rightarrow y(x) = c_1 \cosh(\sqrt{\lambda} x) + c_2 \sinh(\sqrt{\lambda} x)$

BC's \Rightarrow ~~exp exp~~ $y(0) = 0 \Rightarrow c_1 = 0$

So $y(x) = c_2 \sinh(\sqrt{\lambda} x)$

$y'(\pi) = 0 \Rightarrow 0 = c_2 \sqrt{\lambda} \cosh(\sqrt{\lambda} \pi)$

So $\sqrt{\lambda} \neq 0$, $c_2 \neq 0$ if we want non-trivial solutions.

$\Rightarrow 0 = \cosh(\sqrt{\lambda} \pi)$ but no real solutions exist

$\Rightarrow c_2$ must be zero, so only trivial solution $y(x) = 0$

18) $y'' + \lambda y = 0, y'(0) = 0, y'(L) = 0$

Solution: $r = \pm \sqrt{-\lambda}$ as in #14

Consider 3 cases: $\lambda = 0, \lambda > 0, \lambda < 0$

$\lambda = 0$ | $y(x) = c_1 + c_2 x$ and $y'(x) = c_2$
Both BC's imply $c_2 = 0 \Rightarrow \boxed{y(x) = c_1}$ (any constant)

$\lambda > 0$ | $y(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$
 $y'(x) = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda} x) + c_2 \sqrt{\lambda} \cos(\sqrt{\lambda} x)$

BC 1: $y'(0) = 0 \Rightarrow 0 = c_2 \sqrt{\lambda} \Rightarrow c_2 = 0$

BC 2: $y'(L) = 0 \Rightarrow 0 = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda} L)$

For non-trivial solutions, $c_1 \neq 0, \lambda > 0$

$\Rightarrow \sin(\sqrt{\lambda} L) = 0, \text{ so } \sqrt{\lambda} L = n\pi \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2, n \in \mathbb{Z}$

So $\boxed{y(x) = c_1 \cos\left(\frac{n\pi x}{L}\right)}$

$\lambda < 0$ | $y(x) = c_1 \cosh(\sqrt{\lambda} x) + c_2 \sinh(\sqrt{\lambda} x)$
 $y'(x) = c_1 \sqrt{\lambda} \sinh(\sqrt{\lambda} x) + c_2 \sqrt{\lambda} \cosh(\sqrt{\lambda} x)$

BC 1: $y'(0) = 0 \Rightarrow 0 = c_2 \sqrt{\lambda} \Rightarrow c_2 = 0$

BC 2: $y'(L) = 0 \Rightarrow 0 = c_1 \sqrt{\lambda} \sinh(\sqrt{\lambda} L)$

$\sinh(\cdot)$ cannot be zero for positive argument.
 $\Rightarrow c_1 = 0$

Only trivial solution $\boxed{y(x) = 0}$