

HW #6 Solutions

Section 10.5, # 2, 8, 10

Section 10.6, # 8, 15

Section 10.7, # 9

Section 10.5

2) Determine if separation of variables can be applied to the PDE

$$t u_{xx} + x u_t = 0$$

Solution: Assume  $u(x,t) = \phi(x)h(t)$ , then substitution into PDE gives

$$t \phi''(x)h(t) + x \phi(x)h'(t) = 0 \quad \text{Divide by } \phi(x)h(t)xt$$

$$\Rightarrow \frac{\phi''(x)}{x \phi(x)} + \frac{h'(t)}{t h(t)} = 0$$

$$\Rightarrow -\frac{\phi''(x)}{x \phi(x)} = \frac{h'(t)}{t h(t)} = -\lambda \Rightarrow$$

$$\left. \begin{aligned} h'(t) + \lambda t h(t) &= 0 \\ \phi''(x) - \lambda x \phi(x) &= 0 \end{aligned} \right\} \Rightarrow \text{separable.}$$

8) Solve the following initial boundary value problem.

$$4u_t = u_{xx}$$

$$\sum u(0,t) = 0$$

$$\sum u(2,t) = 0$$

$$u(x,0) = 2 \sin\left(\frac{\pi x}{2}\right) - \sin(\pi x) + 4 \sin(2\pi x)$$

Solution: Assume  $u(x,t) = \phi(x)h(t) \Rightarrow \frac{\phi''(x)}{\phi(x)} = 4 \frac{h'(t)}{h(t)} = -\lambda$

$$\Rightarrow \phi''(x)h(t) = 4 \phi(x)h'(t) \quad \text{and} \quad h'(t) = -\frac{1}{4} \lambda h(t)$$

$$\Rightarrow \phi''(x) + \lambda \phi(x) = 0$$

$$u(0,t) = 0 \Rightarrow \phi(0) = 0$$

$$u(2,t) = 0 \Rightarrow \phi(2) = 0$$

Given Dirichlet BC's, only case with non-trivial eigenfunctions is  $\lambda > 0 \Rightarrow r = \pm \sqrt{-\lambda} = \pm i \sqrt{\lambda}$ , so

$$\phi(x) = C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x)$$

$$\text{BC1} \Rightarrow C_1 = 0, \text{ so } \phi(x) = C_2 \sin(\sqrt{\lambda} x)$$

$$\text{BC2} \Rightarrow 0 = C_2 \sin(2\sqrt{\lambda}) \Rightarrow 2\sqrt{\lambda} = n\pi \Rightarrow \sqrt{\lambda} = \frac{n\pi}{2}$$

$$\Rightarrow \phi(x) = C_2 \sin\left(\frac{n\pi x}{2}\right)$$

$$\begin{aligned} \Rightarrow h'(t) + \frac{1}{4} \lambda h(t) &= 0 \\ \Rightarrow e^{\frac{1}{4} \lambda t} h'(t) + \frac{1}{4} \lambda e^{\frac{1}{4} \lambda t} h(t) &= 0 \\ \Rightarrow \frac{d}{dt} \left[ h(t) e^{\frac{1}{4} \lambda t} \right] &= 0 \\ h(t) e^{\frac{1}{4} \lambda t} &= C \\ h(t) &= C e^{-\frac{1}{4} \lambda t} \end{aligned}$$

$$\Rightarrow \phi_n(x) = C_2 \sin\left(\frac{n\pi x}{2}\right), h_n(t) = C e^{-\frac{1}{4}\left(\frac{n\pi}{2}\right)^2 t}$$

$$\Rightarrow u_n(x,t) = C_n \sin\left(\frac{n\pi x}{2}\right) e^{-\frac{1}{4}\left(\frac{n\pi}{2}\right)^2 t}$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{2}\right) e^{-\frac{1}{4}\left(\frac{n\pi}{2}\right)^2 t} \quad \text{now find } B_n$$

$$\Rightarrow u(x,0) = 2 \sin\left(\frac{\pi x}{2}\right) - \sin(\pi x) + 4 \sin(2\pi x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{2}\right)$$

$$\Rightarrow B_1 = 2, B_2 = -1, B_4 = 4 \quad \text{all other } B_n = 0$$

(n=1)            (n=2)            (n=4)

$$\Rightarrow u(x,t) = 2 \sin\left(\frac{\pi x}{2}\right) e^{-\frac{\pi^2}{16} t} - \sin(\pi x) e^{-\frac{\pi^2}{4} t} + 4 e^{-\pi^2 t} \sin(2\pi x)$$

10) Consider the conduction of heat in a rod 40cm in length whose ends are maintained at 0°C for all t > 0. Suppose  $\alpha^2 = 1$  and  $u(x,0) = \begin{cases} x & 0 \leq x < 20 \\ 40-x & 20 \leq x \leq 40 \end{cases}$

Solution:  $L = 40, u(0,t) = u(40,t) = 0, \alpha^2 = 1$  from the directions. Assume  $u(x,t) = \phi(x)h(t)$ , then we get the same result as #8 except the 4 is a 1

$$\Rightarrow \phi''(x) + \lambda \phi(x) = 0 \quad \text{and} \quad h'(t) = -\lambda h(t)$$

Since we have Dirichlet BC's like in #8, the only non-trivial solution to the  $\phi$  ODE, and  $h(t)$  is done identically to #8, so

$$\phi_n(x) = C_2 \sin\left(\frac{n\pi x}{40}\right), \sqrt{\lambda} = \frac{n\pi}{40}, h_n(t) = C e^{-\left(\frac{n\pi}{40}\right)^2 t}$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{40}\right) e^{-\left(\frac{n\pi}{40}\right)^2 t}$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \Rightarrow b_n = \frac{2}{20} \int_0^{20} x \sin\left(\frac{n\pi x}{40}\right) dx + \frac{2}{20} \int_{20}^{40} (40-x) \sin\left(\frac{n\pi x}{40}\right) dx$$

$$\Rightarrow b_n = \frac{2}{20} \left[ -\frac{40(n\pi x \cos\left(\frac{n\pi x}{40}\right) - 40 \sin\left(\frac{n\pi x}{40}\right))}{n^2 \pi^2} \right]_0^{20} + \frac{2}{20} \left[ \frac{40(n\pi(x-40) \cos\left(\frac{n\pi x}{40}\right) - 40 \sin\left(\frac{n\pi x}{40}\right))}{n^2 \pi^2} \right]_{20}^{40}$$

$$= -\frac{40}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \frac{80}{n^2 \pi^2} \sin\left(\frac{n\pi}{2}\right) + \frac{40}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \frac{80}{n^2 \pi^2} \sin\left(\frac{n\pi}{2}\right) - \frac{80}{n^2 \pi^2} \sin(n\pi)$$

$$\Rightarrow b_n = \frac{160}{n^2 \pi^2} \sin\left(\frac{n\pi}{2}\right) \Rightarrow u(x,t) = \frac{160}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n^2} \sin\left(\frac{n\pi x}{40}\right) e^{-\left(\frac{n\pi}{40}\right)^2 t}$$

Section 10.6

8) Find the steady state solution of  $u_t = \alpha^2 u_{xx}$ , for the boundary conditions:  $u(0,t) = T$ ,  $u_x(L,t) + u(L,t) = 0$ .

Solution: For steady state,  $t \rightarrow \infty \Rightarrow u_t = 0$ , so problem reduces to  $u_{xx} = 0$ , which is an ODE given by

$\Rightarrow v''(x) = 0 \Rightarrow v(x) = c_1 + c_2 x$

$u(0,t) = T \Rightarrow v(0) = T$  BC 1

$u_x(L,t) + u(L,t) = 0 \Rightarrow v'(L) + v(L) = 0$  BC 2

BC 1  $\Rightarrow v(0) = T = c_1 \Rightarrow c_1 = T$ , so  $v(x) = T + c_2 x$   
 $v'(x) = c_2$

$\Rightarrow v'(x) + v(x) = c_2 + T + c_2 x = T + c_2(1+x)$

BC 2  $\Rightarrow v'(L) + v(L) = 0 = T + c_2(1+L)$

$\Rightarrow c_2 = -\frac{T}{1+L}$

$\Rightarrow v(x) = T + \left(-\frac{T}{1+L}\right)x \Rightarrow v(x) = \frac{T(1+L-x)}{1+L}$  BC 2

15) Show that the solution to  $\begin{cases} u_t = \alpha^2 u_{xx} & 0 \leq x \leq L \\ u(x,0) = f(x), u(0,t) = 0, u_x(L,t) = 0 \end{cases}$

is given by  $u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t)$ , where  $u_n(x,t) = \exp\left(-\frac{(2n-1)^2 \pi^2 \alpha^2}{4L^2} t\right) \sin\left(\frac{(2n-1)\pi x}{2L}\right)$

Solution: Assume  $u(x,t) = \phi(x)h(t)$  BC 1  $u(0,t) = 0 \Rightarrow \phi(0) = 0$   
BC 2  $u_x(L,t) = 0 \Rightarrow \phi'(L) = 0$

$\Rightarrow \alpha^2 \phi''(x)h(t) = \phi(x)h'(t)$

$\Rightarrow \frac{\phi''(x)}{\phi(x)} = \frac{1}{\alpha^2} \frac{h'(t)}{h(t)} = -\lambda \Rightarrow \begin{cases} \phi''(x) + \lambda \phi(x) = 0 \\ h'(t) = -\alpha^2 \lambda h(t) \end{cases}$

$r^2 + \lambda = 0 \Rightarrow r = \pm \sqrt{-\lambda}$

For  $\phi(x)$ :  $\lambda = 0$   $\Rightarrow \phi(x) = c_1 + c_2 x$  BC 1  $\Rightarrow c_1 = 0 \Rightarrow$  only trivial.  
BC 2  $\Rightarrow c_2 = 0$

$\lambda < 0$   $\Rightarrow \phi(x) = c_1 \cosh(\sqrt{\lambda} x) + c_2 \sinh(\sqrt{\lambda} x)$  BC 1  $\Rightarrow c_1 = 0$   
 $\phi'(x) = c_1 \sqrt{\lambda} \sinh(\sqrt{\lambda} x) + c_2 \sqrt{\lambda} \cosh(\sqrt{\lambda} x)$  BC 2  $\Rightarrow 0 = c_2 \sqrt{\lambda} \cosh(\sqrt{\lambda} L)$   
 must have  $c_2 = 0 \Rightarrow$  only trivial

$\lambda > 0$   $\Rightarrow \phi(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$  BC 1  $\Rightarrow c_1 = 0$   
 $\phi'(x) = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda} x) + c_2 \sqrt{\lambda} \cos(\sqrt{\lambda} x)$  BC 2  $\Rightarrow 0 = c_2 \sqrt{\lambda} \cos(\sqrt{\lambda} L)$   
 $\Rightarrow \sqrt{\lambda} L = (n - \frac{1}{2})\pi$   
 $\Rightarrow \sqrt{\lambda} = \frac{(n - \frac{1}{2})\pi}{L}$

$$\Rightarrow h'(t) = -\alpha^2 \lambda h(t) \Rightarrow h(t) = C e^{-\alpha^2 \lambda t}$$

$$\Rightarrow h(t) = C e^{-\alpha^2 \left(\frac{(n-\frac{1}{2})^2 \pi^2}{L^2}\right) t}$$

$$\Rightarrow u_n(x,t) = \exp\left(-\frac{\alpha^2 (n-\frac{1}{2})^2 \pi^2}{L^2} t\right) \sin\left(\frac{(n-\frac{1}{2})\pi}{L} x\right)$$

$$\text{and } c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{(n-\frac{1}{2})\pi x}{L}\right) dx$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) \quad \text{where } c_n, u_n(x,t) \text{ are given above.}$$

## Section 10.7

9) From the problem description, we deduce to solve the initial boundary value problem

$$u_{tt} = \alpha^2 u_{xx}$$

$$\text{BC's } \Rightarrow \begin{cases} u(0,t) = 0 \\ u_x(L,t) = 0 \end{cases}$$

$$\text{IC's } \Rightarrow \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = 0 \end{cases}$$

Solution: Assume  $u(x,t) = \phi(x)h(t)$

$$\Rightarrow h''(t)\phi(x) = \alpha^2 h(t)\phi''(x) \Rightarrow \frac{\phi''(x)}{\phi(x)} = \frac{1}{\alpha^2} \frac{h''(t)}{h(t)} = -\lambda$$

$$\Rightarrow \phi''(x) + \lambda \phi(x) = 0 \quad \text{and } h''(t) + \alpha^2 \lambda h(t) = 0$$

The only nontrivial case is  $\lambda > 0$

$$\Rightarrow \phi(x) = c_2 \sin\left(\frac{(n-\frac{1}{2})\pi x}{L}\right); \sqrt{\lambda_n} = \frac{(n-\frac{1}{2})\pi}{L}$$

(Same as #15, 10.6; check this yourself!)

$$\Rightarrow h''(t) + \alpha^2 \lambda_n h(t) = 0 \quad \lambda_n = \left(\frac{(n-\frac{1}{2})\pi}{L}\right)^2$$

$$\Rightarrow h(t) = c_1 \cos\left(\frac{\alpha(n-\frac{1}{2})\pi}{L} t\right) + c_2 \sin\left(\frac{\alpha(n-\frac{1}{2})\pi}{L} t\right)$$

Since we know  $\lambda > 0$  is the only case from  $\phi$  problem.

$$\Rightarrow u_n(x,t) = \phi_n(x)h_n(t) = c_1 \cos\left(\frac{\alpha(n-\frac{1}{2})\pi}{L} t\right) \sin\left(\frac{(n-\frac{1}{2})\pi x}{L}\right) + c_2 \sin\left(\frac{\alpha(n-\frac{1}{2})\pi}{L} t\right) \sin\left(\frac{(n-\frac{1}{2})\pi x}{L}\right)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{\alpha(n-\frac{1}{2})\pi}{L} t\right) \sin\left(\frac{(n-\frac{1}{2})\pi x}{L}\right) + B_n \sin\left(\frac{\alpha(n-\frac{1}{2})\pi}{L} t\right) \sin\left(\frac{(n-\frac{1}{2})\pi x}{L}\right)$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{(n-\frac{1}{2})\pi x}{L}\right)$$

$$u_t(x,0) = g(x) = 0 = \sum_{n=1}^{\infty} \left(\frac{\alpha(n-\frac{1}{2})\pi}{L}\right) B_n \sin\left(\frac{(n-\frac{1}{2})\pi x}{L}\right) \Rightarrow B_n = 0$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{\alpha(n-\frac{1}{2})\pi}{L} t\right) \sin\left(\frac{(n-\frac{1}{2})\pi x}{L}\right); \quad A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{(n-\frac{1}{2})\pi x}{L}\right) dx$$