

MATH 146B

HW #7 Solutions

Section 10.8: #2, 5, 7

2) Find the solution $u(x,y)$ of Laplace's equation on a rectangle $0 < x < a, 0 < y < b$, that satisfies

$$\begin{aligned} u(0,y) &= 0 & u(a,y) &= 0 & 0 < y < b \\ u(x,0) &= h(x) & u(x,b) &= 0 & 0 \leq x \leq a \end{aligned}$$

Solution: Assume $u(x,y) = \phi(x)\psi(y)$, and $\Delta u = 0$ or $u_{xx} + u_{yy} = 0$

$$\begin{aligned} \Rightarrow \phi''(x)\psi(y) + \phi(x)\psi''(y) &= 0 \\ \Rightarrow \frac{\phi''(x)}{\phi(x)} &= -\frac{\psi''(y)}{\psi(y)} = -\lambda \quad \Rightarrow \begin{cases} \phi''(x) + \lambda\phi(x) = 0 & w/ \phi(0)=0, \phi(a)=0 \\ \psi''(y) - \lambda\psi(y) = 0 & w/ \psi(b)=0 \end{cases} \end{aligned}$$

Note that ϕ is the usual ODE eigenvalue problem with Dirichlet boundary conditions, so $\phi_n(x) = c_n \sin\left(\frac{n\pi x}{a}\right)$ with $\sqrt{\lambda_n} = \frac{n\pi}{a}$

Solving that problem, we found $\lambda > 0$ gives non-trivial eigenfunctions. So since ψ has $-\lambda$ in the ODE, then the cases are reversed, so

$$\psi_n(y) = d_1 \cosh\left(\frac{n\pi}{a}(b-y)\right) + d_2 \sinh\left(\frac{n\pi}{a}(b-y)\right)$$

where we have used the fact that the PDE is translation invariant. So we can shift the argument so plugging in Boundary conditions is

easy. The boundary condition is $\psi(b) = 0$

$$\Rightarrow d_1 \cosh(0) + d_2 \sinh(0) = d_1 = 0 \Rightarrow d_1 = 0$$

$$\Rightarrow \psi_n(y) = d_2 \sinh\left(\frac{n\pi}{a}(b-y)\right)$$

$$\Rightarrow u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi}{a}(b-y)\right)$$

Last BC $\Rightarrow u(x,0) = h(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi b}{a}\right)$

$$\Rightarrow B_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a} \int_0^a h(x) \sin\left(\frac{n\pi x}{a}\right) dx \Rightarrow B_n = \frac{\frac{2}{a} \int_0^a h(x) \sin\left(\frac{n\pi x}{a}\right) dx}{\sin\left(\frac{n\pi b}{a}\right)}$$

5) Find the solution to Laplace's equation

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

outside $r=a$, $0 \leq \theta < 2\pi$, $u(a, \theta) = f(\theta)$ BC.

Solution: Assume $u(r, \theta) = \phi(\theta) G(r)$. Plug into PDE

$$\Rightarrow G''(r) \phi(\theta) + \frac{1}{r} G'(r) \phi(\theta) + \frac{1}{r^2} \phi''(\theta) G(r) = 0$$

Divide by $\frac{1}{r^2} G(r) \phi(\theta)$

$$\Rightarrow r^2 \frac{G''(r)}{G(r)} + r \frac{G'(r)}{G(r)} + \frac{\phi''(\theta)}{\phi(\theta)} = 0$$

$$\Rightarrow r^2 \frac{G''(r)}{G(r)} + r \frac{G'(r)}{G(r)} = -\frac{\phi''(\theta)}{\phi(\theta)} = \lambda$$

$$\Rightarrow r^2 G''(r) + r G'(r) - \lambda G(r) = 0$$

$$\Rightarrow r^2 G''(r) + r G'(r) - n^2 G(r) = 0 \text{ from } \lambda = n^2 \text{ since Euler ODE}$$

$$\text{Let } G(r) = r^p, \text{ since Euler ODE}$$

$$\Rightarrow [p(p-1) + p - n^2] r^p = 0$$

$$\Rightarrow p = \pm n, \text{ and assuming } n \neq 0$$

$$\Rightarrow G(r) = C_1 r^n + C_2 r^{-n}, \text{ but note}$$

$r^n \rightarrow \infty$ as $n \rightarrow \infty$ since we are outside the circle $r=a$

$$\Rightarrow G(r) = C_2 r^{-n}$$

Now combining information,

$$u(r, \theta) = \frac{C_0}{2} + \sum_{n=1}^{\infty} r^{-n} (A_n \cos(n\theta) + B_n \sin(n\theta))$$

$$\text{BC } \Rightarrow u(a, \theta) = \frac{C_0}{2} + \sum_{n=1}^{\infty} a^{-n} (A_n \cos(n\theta) + B_n \sin(n\theta))$$

$$\Rightarrow \left[\frac{A_n}{a^n} = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos(n\theta) d\theta, \quad \frac{B_n}{a^n} = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin(n\theta) d\theta \right]$$

7) Find the solution to Laplace's equation in a circular sector

$0 < r < a$, $0 < \theta < \alpha$ such that

$$u(r, 0) = 0, \quad u(r, \alpha) = 0 \quad 0 \leq r \leq a$$

$$u(a, \theta) = f(\theta) \quad 0 \leq \theta \leq \alpha$$

Assume u is single valued and bounded, and $0 < \alpha < 2\pi$

Solution: Assume $u(r, \theta) = G(r) \phi(\theta)$ and $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$

$$\Rightarrow r^2 \frac{G''(r)}{G(r)} + r \frac{G'(r)}{G(r)} = - \frac{\phi''(\theta)}{\phi(\theta)} = \lambda \quad \text{as in #5}$$

$$\Rightarrow \phi''(\theta) + \lambda \phi(\theta) = 0 \quad \xrightarrow{\text{From BC's}} \phi(\theta) = C_1 \cos(\sqrt{\lambda} \theta) + C_2 \sin(\sqrt{\lambda} \theta)$$

$$r^2 G''(r) + r G'(r) - \lambda G(r) = 0 \quad \xrightarrow{\text{and}} \phi(\theta) = C_2 \sin\left(\frac{n\pi\theta}{\alpha}\right), \sqrt{\lambda} = \frac{n\pi}{\alpha}$$

\hookrightarrow Euler DE, so $G(r) = d_1 r^{\frac{n\pi}{\alpha}} + d_2 r^{-\frac{n\pi}{\alpha}}$, but here, we are in the sector, so $r^{-\frac{n\pi}{\alpha}} \rightarrow 0$ as $r \rightarrow 0 \Rightarrow G(r) = d_1 r^{\frac{n\pi}{\alpha}}$

$$\Rightarrow u(r, \theta) = \sum_{n=1}^{\infty} B_n r^{\frac{n\pi}{\alpha}} \sin\left(\frac{n\pi\theta}{\alpha}\right), \text{ with}$$

$$\frac{B_n}{r^{\frac{n\pi}{\alpha}}} = \frac{2}{\alpha} \int_0^\alpha f(\theta) \sin\left(\frac{n\pi\theta}{\alpha}\right) d\theta$$