

Chapter 5 - Review

①

Make sure that you review the basics about series, the material included in 9C (this is reviewed in Section 5.1).

* Ratio Test, Root Test, Taylor Series, etc.

For 5.2, series solutions are

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Keys to these questions:

* Be sure to review re-indexing series

* Solve for recurrence relations for the highest sub indices

(e) $\underline{a_n} = \frac{a_{n-1}}{n!}$ $n > n-1$ for example.

Euler ODEs

$$x^2 y'' + \alpha x y' + \beta y = 0 \quad \alpha, \beta \text{ constants}$$

is the most basic example. You can easily solve these using $y = x^r$ as a substitution and finding r .

- ① r distinct real roots $\Rightarrow y = C_1 x^{r_1} + C_2 x^{r_2}$
② repeated real roots $\Rightarrow y = (C_1 + C_2 \ln x) x^r$
③ complex roots $\Rightarrow y = C_1 x^\lambda \cos(u \ln x) + C_2 x^\lambda \sin(u \ln x)$
 $r = \lambda \pm i u$

$$\text{If you have } P(x) y'' + Q(x) y' + R(x) y = 0$$

If x_0 is a regular singular point if

$$\lim_{x \rightarrow x_0} (x-x_0) \frac{Q(x)}{P(x)} < \infty$$

$$\lim_{x \rightarrow x_0} (x-x_0)^2 \frac{R(x)}{P(x)} < \infty$$

hold.

From section 5.5 on, we use the substitutions

$$y = \sum_{n=0}^{\infty} a_n x^{r+n}, \quad y' = \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1}, \quad y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n-2}$$

Note: These summations start at $n=0$.

The "r" are the exponents at singularity
the coeffs of a_0 are the indicial equation

Alternatively, if ODE is $x^2 y'' + \alpha x y' + \beta y = 0$,

then the indicial eqn is $r(r-1) + \alpha r + \beta = 0$

Suggested Questions to Look over and work through:

Section 5.5 - #1, 3, 4

Section 5.6 - #14

Section 5.7 - #1

* The best way to do these is to look through the solution and understand how to go from one step to the next. Then try doing the question yourself without the solution.

Chapter 10 Review

(2)

Eigenvalue Problem

$$y'' + \lambda y = 0 \quad (\text{Note:})$$

$$\lambda > 0 \Rightarrow r = \pm i\sqrt{\lambda} \Rightarrow y(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$$

$$\lambda = 0 \Rightarrow 0, 0 \Rightarrow y(x) = C_1 x + C_2$$

$$\lambda < 0 \Rightarrow r = \pm \sqrt{\lambda} \Rightarrow y(x) = C_1 \cosh(\sqrt{\lambda}x) + C_2 \sinh(\sqrt{\lambda}x)$$

Note: For $y'' - \lambda y = 0$ the cases $\lambda > 0, \lambda < 0$ are reversed.

Summary of Boundary Conditions

$$\text{For } \phi''(x) + \lambda \phi(x) = 0$$

B.C.'s	$\phi(0) = 0$ $\phi(L) = 0$	$\phi'(0) = 0$ $\phi'(L) = 0$	$\phi(0) = 0$ $\phi'(L) = 0$	$\phi'(0) = 0$ $\phi(L) = 0$
Eigenvalues	$(\frac{n\pi}{L})^2$	$(\frac{n\pi}{L})^2$	$(\frac{(n-1/2)\pi}{L})^2$	$(\frac{(n-1/2)\pi}{L})^2$
Eigenfunctions	$\sin(\frac{n\pi x}{L})$	$\cos(\frac{n\pi x}{L})$	$\sin(\frac{(n-1/2)\pi x}{L})$	$\cos(\frac{(n-1/2)\pi x}{L})$

Fourier Series Formula

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

$$A_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \quad B_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Note: Know how to draw the functions (periodic). See HW Sec 10.2

Fourier Sine/Cosine Series

Cosine Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

Sine Series

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Note: Know how to sketch Fourier sine/cosine series.

Separation of Variables

Use product solution $u(x,t) = \phi(x)h(t)$ and sub into PDE

Heat Equation: $u_t = \alpha^2 u_{xx}$

Wave Equation: $u_{tt} = a^2 u_{xx}$

Laplace Equation: $u_{xx} + u_{yy} = 0$ or $\Delta u = 0$

Heat Equation

Usual Heat equation is solved using sep. of variables.

* See Sec 10.5 # 8, 9 for examples

Steady State and Transient (Homogeneous) Solution

- This is for when you have non-homogeneous BC's

PDE problem: $u_t = u_{xx}$ with BC's $u(0,t) = T_1$
 $u(L,t) = T_2$
 $u(x,0) = f(x)$

would be provided.

Steady State Soln: $v(x) = (T_2 - T_1) \frac{x}{L} + T_1$

Let $u(x,t) = v(x) + w(x,t)$

$w(x,t)$ satisfies $w_t = w_{xx}$ and

New BC's $\begin{cases} w(0,t) = u(0,t) - v(0) \\ w(L,t) = u(L,t) - v(L) \end{cases}$

New IC's $\begin{cases} w(x,0) = u(x,0) - v(x) \end{cases}$

Wave Equation

D'Alembert's Solution

Wave equation ~~u_{tt} = a^2 u_{xx}~~ $u_{tt} = a^2 u_{xx}$

IC's $u(x,0) = f(x), u_t(x,0) = g(x)$

$$\text{Then } u(x,t) = \frac{1}{2} [f(x-at) + f(x+at)] + \frac{1}{2a} \int_{x-at}^{x+at} g(\xi) d\xi$$

For a review of separating variables, see #1,2,3 in 10.5

Suggested Problems

Section 10.1: All assigned (These are good practice for the rest of chapter 10)

Section 10.2: #13, 14

Section 10.3: #4

Section 10.4: #7, 16, 17

Section 10.5: #8, 9

Section 10.6: #1, 3

Additional PDF Question * Know this question!

