

Laplace's Equation

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From Boyce + Diprima Section 10.8

2) $\Delta u = u_{xx} + u_{yy} = 0$

$$u(0, y) = 0 \quad u(x, 0) = h(x)$$

$$u(a, y) = 0 \quad u(x, b) = 0$$

Solution: Assume a product solution: $u(x, y) = \phi(x) \psi(y)$

Now use separation of variables:

$$\phi''(x) \psi(y) + \phi(x) \psi''(y) = 0$$

$$\Rightarrow \frac{\phi''(x)}{\phi(x)} = -\frac{\psi''(y)}{\psi(y)} = -\lambda$$

$$\Rightarrow \phi''(x) + \lambda \phi(x) = 0 \quad \text{and} \quad \psi''(y) - \lambda \psi(y) = 0$$

The ODE in terms of x is the usual eigenvalue problem.
We work with this one first since its BC's are homogeneous

$$\begin{aligned} u(0, y) = 0 &\Rightarrow \phi(0) = 0 \Rightarrow \phi(x) = C_1 \cos \sqrt{\lambda} x + C_2 \sin \sqrt{\lambda} x \\ u(a, y) = 0 &\Rightarrow \phi(a) = 0 \Rightarrow \phi(x) = C_2 \sin \left(\frac{n\pi x}{a} \right), \sqrt{\lambda} = \frac{n\pi}{a} \end{aligned}$$

For the ODE in terms of y , we have to use the fact that
Laplace's Equation is translation invariant to solve this
problem. Since we have $-\lambda$ in $\psi'' - \lambda \psi = 0$, we will
get $\psi(y) = d_1 \cosh \left(\frac{n\pi}{a} y \right) + d_2 \sinh \left(\frac{n\pi}{a} y \right)$
Since Δu is translation invariant (shifting y by a constant)
the translated ψ will still satisfy the PDE
 $\Rightarrow \psi(y) = d_1 \cosh \left(\frac{n\pi}{a} (b-y) \right) + d_2 \sinh \left(\frac{n\pi}{a} (b-y) \right)$
also solves the ODE.



The condition $U(x,b) = 0 \Rightarrow \psi(b) = 0$ This is why we need to shift our solution, so the boundary conditions work. So then you get that

$$\psi(y) = c_2 \sinh\left(\frac{n\pi}{a}(b-y)\right)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi}{a}(b-y)\right) \quad \star$$

Now the last BC will give us A_n formula, since

$$u(x,0) = h(x) = \sum_{n=1}^{\infty} \underbrace{A_n \sinh\left(\frac{n\pi b}{a}\right)}_{(*)} \sin\left(\frac{n\pi x}{a}\right)$$

Note that $(*)$ is a constant since $A_n, \sin\left(\frac{n\pi b}{a}\right)$ are constant. So this is just the Fourier Sine Series for $h(x)$. So as before with the wave equation, we set all of $(*)$ equal to the coefficient formula,

$$A_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a} \int_0^a h(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

$$\Rightarrow A_n = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a h(x) \sin\left(\frac{n\pi x}{a}\right) dx \quad (+)$$

and \star is the solution with A_n defined as $(+)$



It may be pertinent to attempt additional questions from 10.8 to practice your skills solving these PDE for the final.