

Wave Equation

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Before I solve the problem, I will solve the wave equation problem in a general setting to show the method. It will be helpful if you can solve this general PDE on your own to get a good understanding of the technique. Then look at the assigned question and work through that.

Here is the general initial boundary value problem

$$u_{tt} = u_{xx} \quad \text{for } x \in [0, L], t > 0$$

$$\text{BC's} \begin{cases} u(0, t) = 0 \\ u(L, t) = 0 \end{cases}$$

$$\text{IC's} \begin{cases} u(x, 0) = f(x) \\ u_t(x, 0) = g(x) \end{cases}$$

Solution: Use separation of variables, $u(x, t) = \phi(x)h(t)$.

$$\Rightarrow \phi(x)h''(t) = \phi''(x)h(t)$$

$$\Rightarrow \frac{\phi''(x)}{\phi(x)} = \frac{h''(t)}{h(t)} = -\lambda \quad \begin{matrix} \text{(We choose } -\lambda \\ \text{for convenience in the} \\ \text{calculation)} \end{matrix}$$

$$\Rightarrow \phi''(x) + \lambda \phi(x) = 0 \quad \text{and} \quad h''(t) + \lambda h(t) = 0$$

$$\text{The BC's become } \begin{cases} \phi(0) = 0 \\ \phi(L) = 0 \end{cases}$$

as before w/ the heat equation

$\lambda > 0$ yields only non-trivial solutions
(you should check this!!)

$$\Rightarrow \phi(x) = C_2 \sin \sqrt{\lambda} x \quad \text{with } \sqrt{\lambda} = \frac{n\pi}{L} \text{ as before}$$

$$\Rightarrow \phi(x) = C_2 \sin \left(\frac{n\pi x}{L} \right)$$

Here, we have an eigenvalue problem in h as well. We expect oscillating solns for wave eqn

$$\Rightarrow h(t) = C_1 \cos \sqrt{\lambda} t + C_2 \sin \sqrt{\lambda} t$$

$$\sqrt{\lambda} = \frac{n\pi}{L}$$

So now we have $\phi(x)$ and $h(t)$.



Recall $u(x,t) = \phi(x) h(t)$

$$\Rightarrow = \tilde{C}_1 \cos \sqrt{\lambda} t \sin \sqrt{\lambda} x + \tilde{C}_2 \sin \sqrt{\lambda} t \sin \sqrt{\lambda} x$$

$$\textcircled{*} \Rightarrow u(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi t}{L}\right) \sin\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi t}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

Now, all that's left is to find A_n and B_n using the IC's given at the beginning. So, plugging in 0 for t

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \quad \left. \begin{array}{l} \text{exactly Fourier} \\ \text{sine series for } f(x) \end{array} \right\} (*)$$

Taking the derivative of $\textcircled{*}$ wrt. t , we get

$$u_t(x,t) = \sum_{n=1}^{\infty} -A_n \frac{n\pi}{L} \sin\left(\frac{n\pi t}{L}\right) \sin\left(\frac{n\pi x}{L}\right) + B_n \frac{n\pi}{L} \cos\left(\frac{n\pi t}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

So then, plugging $t = 0$,

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi}{L} \sin\left(\frac{n\pi x}{L}\right) \quad \left. \begin{array}{l} \text{exactly Fourier} \\ \text{sine series for } g(x) \end{array} \right\} (**)$$

So then the formulas for the coefficients are

$$\text{coeffs} \left[\begin{array}{l} A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \\ B_n \frac{n\pi}{L} = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad (\text{Note we take } \underline{\underline{a}} \text{ of } (t) \text{ from above}) \end{array} \right]$$

So the solution is $\textcircled{*}$ with A_n and B_n defined as above. If you were given $f(x)$ and $g(x)$ explicitly, we could solve for B_n and A_n above by doing the integration



Problem: $u_{tt} = u_{xx}$ for $x \in [0, 1]$, $t > 0$ (2)

$$\text{BC's } \begin{cases} u(0, t) = 0 \\ u(1, t) = 0 \end{cases}$$

$$\text{IC's } \begin{cases} u(x, 0) = x - x^2 \\ u_t(x, 0) = 0 \end{cases}$$

Solution: Letting $u(x, t) = \phi(x)h(t)$ as before, you should get

$$\phi''(x) + \lambda \phi(x) = 0 \quad \text{and} \quad h''(t) + \lambda h(t) = 0$$

Solving the above using BC's and $L = 1$, you will find

$$\phi(x) = C_2 \sin(n\pi x), \sqrt{\lambda} = n\pi, h(t) = C_1 \cos(n\pi t) + C_2 \sin(n\pi t)$$

$$\Rightarrow u(x, t) = \sum_{n=1}^{\infty} A_n \cos(n\pi t) \sin(n\pi x) + B_n \sin(n\pi t) \sin(n\pi x) \quad (\star)$$

$$u_t(x, t) = \sum_{n=1}^{\infty} -A_n n\pi \sin(n\pi t) \sin(n\pi x) + B_n n\pi \cos(n\pi t) \sin(n\pi x)$$

As we did previously, so now using the IC's,

$$u(x, 0) = x - x^2 = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \quad (1)$$

$$u_t(x, 0) = 0 = \sum_{n=1}^{\infty} B_n n\pi \sin(n\pi x) \quad (2)$$

Fourier sine series
for $x - x^2$

(2) $\Rightarrow B_n = 0$, see formula for coeffs on previous page.

$$(1) \Rightarrow A_n = 2 \int_0^1 (x - x^2) \sin(n\pi x) dx = 2 \int_0^1 x \sin(n\pi x) dx - 2 \int_0^1 x^2 \sin(n\pi x) dx$$

$$\Rightarrow A_n = \frac{2}{n^2\pi^2} \sin(n\pi) - \left[\frac{1}{n\pi} \cos(n\pi) \right] + \left[\frac{4}{n^3\pi^3} \cos(n\pi) \right] + \left[\frac{2}{n\pi} \cos(n\pi) \right]$$

$$-\left[\frac{4}{n^3\pi^2} \sin(n\pi) \right] + \left[\frac{4}{n^3\pi^3} \right] \quad \text{by integration by parts}$$

$$\Rightarrow A_n = -\frac{1}{n\pi} (-1)^n - \frac{4}{n^3\pi^3} (-1)^n + \frac{4}{n^3\pi^3} = -\frac{1}{n\pi} (-1)^n + \frac{4}{n^3\pi^3} ((-1)^n - 1)$$

So answer is (\star) with $B_n = 0$ and A_n as above. ■

It is recommended that you attempt more of
these types of questions to solidify your skills
on these kinds of problems.