Last Name: First Name:

MATH 65B - Spring 2018

Groupwork 10: April 12, 2018

1. Find the general term a_n of the sequence:

(a) $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \ldots\right\}$

(b) $\{2, 7, 12, 17\}$

(c) $\left\{1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \ldots\right\}$

(d) $\left\{\sin\left(\frac{\pi}{2}\right), \sin\left(\frac{3\pi}{2}\right), \sin\left(\frac{5\pi}{2}\right), \sin\left(\frac{7\pi}{2}\right), \ldots\right\}$

Solution:

(a) Note that all the numerators are equal to 1, and all the denominators are even numbers. Recall that all even numbers can be represented by $2n$ for $n = 1, 2, 3, \ldots$. Therefore, we can write out the general term as

$$a_n = \frac{1}{2n} \qquad \text{for } n = 1, 2, 3, 4, \ldots$$

(b) This is an arithmetic sequence, so you can use the formula $a_n = a_1 + (n-1)d$. We want to move away from formula memorization, so we can just figure out the pattern. Note that the first term is 2, and we add 5 to the previous term to get the new term. So for $n = 1$, we have $a_1 = 2$. For $n = 2$ we add 5 (or 1(5)) to a_1. For $n = 3$ we add 10 (or 2(5)) to a_1; for $n = 4$ we add 15 (or 3(5)) to a_1, and so on. Notice that the number being multiplied by 5 is always one less than the n we are computing. Thus we get

$$a_n = 2 + 5(n-1) \qquad \text{for } n = 1, 2, 3, 4, \ldots$$

(c) For a more complicated sequence, recall that we can usually deal with them by looking at the numerator, denominator, and sign separately, then put the pieces together. Notice that the sign pattern is $+, -, +, -, \ldots$, so if we start at $n = 0$, then there must be a $(-1)^n$ term present. The sequence of just numerators is $1, 2, 4, 8, \ldots$, which is the powers of 2, so we get 2^n. The sequence in the denominator is similar, but powers of three, so it is 3^n. Thus we get

$$a_n = (-1)^n \frac{2^n}{3^n} = (-1)^n \left(\frac{2}{3}\right)^n = \left(-\frac{2}{3}\right)^n \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

(d) For this sequence, we only have to figure out the pattern inside of the sine function. There is always 2 in the denominator, and a π in the numerator. The coefficient of the π is changing, and is always an odd integer which can be represented as $2n - 1$ for $n = 1, 2, 3, \ldots$. So we get

$$a_n = \sin\left(\frac{(2n-1)\pi}{2}\right) = \sin\left(\left(n - \frac{1}{2}\right)\pi\right) = (-1)^{n+1} \qquad \text{for } n = 1, 2, 3, \ldots$$

Please, show all work.

2. Determine whether the sequence converges or diverges. If it converges, find its limit.

$$(a) \qquad a_n = n\sin\left(\frac{1}{n}\right)$$

$$(b) \qquad a_n = \left(1+\frac{2}{n}\right)^n$$

$$(c) \qquad a_n = \frac{n!}{2^n}$$

__Solution:__

(a) By rewriting the n in front, and using the substitution $u = \dfrac{1}{n}$, we have

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \\ &= \lim_{u\to 0}\frac{\sin(u)}{u} = 1 \qquad \text{convergent}\end{aligned}$$

(b) For this sequence, notice that the n is in the exponent. From the earlier part of the course on L'Hopital's Rule, recall that we use the exponential-logarithm trick to deal with these cases since we have the indeterminate form 1^∞.

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = \lim_{n\to\infty}\exp\left(\ln\left(\left(1+\frac{2}{n}\right)^n\right)\right) \\ &= \lim_{n\to\infty}\exp\left(n\ln\left(1+\frac{2}{n}\right)\right) = \lim_{n\to\infty}\exp\left(\frac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}}\right) \qquad \text{Use } u=\frac{1}{n} \\ &= \exp\left(\lim_{u\to 0^+}\frac{\ln(1+2u)}{u}\right) \qquad \text{Use L'Hopital's Rule} \\ &= \exp\left(\lim_{u\to 0^+}\frac{\frac{2}{1+2u}}{1}\right) = \exp\left(\lim_{u\to 0^+}\frac{2}{1+2u}\right) \\ &= \exp(2) = e^2\end{aligned}$$

(c) Let's write out the n^{th} term in the sequence and see what we can deduce.

$$a_n = \frac{n!}{2^n} = \frac{1}{2}\cdot\left[\frac{2}{2}\cdot\frac{3}{2}\cdot\frac{4}{2}\cdot\ldots\cdot\frac{(n-1)}{2}\right]\cdot\frac{n}{2} \geq \frac{1}{2}\cdot\frac{n}{2} = \frac{n}{4} \to \infty \quad \text{as} \quad n\to\infty$$

Where the inequality follows from the fact that the terms from $\dfrac{2}{2}$ to $\dfrac{(n-1)}{2}$ multiplied together must be greater than or equal to 1, since all of the fractions themselves are greater than or equal to 1. Therefore the sequence is greater than $\dfrac{n}{4}$, which is divergent, so then a_n is divergent.

Please, show all work.

3. Determine whether the series converges or diverges. If it converges, find its sum. You must show all the steps!

$$\sum_{n=1}^{\infty}\left(\frac{1}{e^n}+\frac{1}{n(n+1)}\right)$$

<u>**Solution:**</u>

First note the the first term is a geometric series, and the second is a telescoping series. We can only break the series into two parts if both of the series are convergent. So we deal with them separately. First the geometric series

$$\sum_{n=1}^{\infty}\frac{1}{e^n}=\sum_{n=1}^{\infty}\left(\frac{1}{e}\right)^n=\sum_{n=1}^{\infty}\frac{1}{e}\left(\frac{1}{e}\right)^{n-1}=\frac{1/e}{1-1/e}=\frac{1}{e-1}\quad \text{convergent since } \frac{1}{e}<1$$

The second series requires that we use the definition of a series, a limit of partial sums. The second series has the general term $a_n=\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$, then the partial sum s_n is

$$\begin{aligned} s_n &= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots+\left(\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &= 1-\frac{1}{n+1}\end{aligned}$$

Then by taking the limit of partial sums, we have

$$\lim_{n\to\infty} 1-\frac{1}{n+1}=1$$

So since both series are convergent, we can break the sum, and we have already computed the sums individually, hence

$$\sum_{n=1}^{\infty}\left(\frac{1}{e^n}+\frac{1}{n(n+1)}\right)=\sum_{n=1}^{\infty}\frac{1}{e^n}+\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\frac{1}{e-1}+1=\frac{e}{e-1}$$

Please, show all work.

4. For (a), determine whether the series converges or diverges. For (b), write the repeating decimal as a ratio of integers (ie. a rational number). For (c), determine for which values of x does the series converges, and find its sum.

$$(a) \qquad \sum_{n=1}^{\infty} \frac{1}{e^n}$$

$$(b) \qquad 4.342342342\ldots$$

$$(c) \qquad \sum_{n=0}^{\infty} \frac{\cos^n(x)}{2^n}$$

Solution:

(a) The series looks like a geometric series, as e is a constant. Recall that the geometric series has the general form

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

So in order to use the formula, we must rewrite the series in one of the general forms above. Since our series in indexed at $n = 1$, we can use the second formula so we don't have to reindex a series

$$\sum_{n=1}^{\infty} \frac{1}{e^n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n = \sum_{n=1}^{\infty} \frac{1}{e}\left(\frac{1}{e}\right)^{n-1} = \frac{\frac{1}{e}}{1-\frac{1}{e}} = \frac{1}{e-1}$$

(b) First we notice that the repeats come after the decimal point, so that we have $4.342342342... = 4.\overline{342}$. Now we are regrouping every three digits

$$\begin{aligned}
4.\overline{342} &= 4 + 0.342 + 0.000342 + 0.000000342 + 0.000000000342 + \ldots \\
&= 4 + \frac{342}{10^3} + \frac{342}{10^6} + \frac{342}{10^9} + \frac{342}{10^12} + \ldots \\
&= 4 + \sum_{n=1}^{\infty} \frac{342}{10^{3n}} = 4 + 342\sum_{n=1}^{\infty}\left(\frac{1}{10^3}\right)^n \\
&= 4 + 342\sum_{n=1}^{\infty}\left(\frac{1}{1000}\right)^n = 4 + 342\sum_{n=1}^{\infty}\frac{1}{1000}\left(\frac{1}{1000}\right)^{n-1} \\
&= 4 + 342\frac{\frac{1}{1000}}{1-\frac{1}{1000}} = 4 + 342\frac{1}{1000}\frac{1000}{999} = 4 + \frac{342}{999} = \frac{3996+342}{990} = \frac{4338}{999}
\end{aligned}$$

(c) Ignore the x in the cosine term for a moment. Now notice that the series is geometric,

$$\sum_{n=0}^{\infty} \frac{\cos^n(x)}{2^n} = \sum_{n=0}^{\infty} \left(\frac{\cos(x)}{2}\right)^n,$$

where we have $r = \dfrac{\cos(x)}{2}$. So geometric series converge for $|r| < 1$, and $|r| = \dfrac{|\cos(x)|}{2} \leq \dfrac{1}{2} < 1$. That inequality says that our r is *always* strictly less than 1, so the series converges for all x. By the formula,

$$\sum_{n=0}^{\infty} \frac{\cos^n(x)}{2^n} = \sum_{n=0}^{\infty} \left(\frac{\cos(x)}{2}\right)^n = \frac{1}{1-\frac{\cos(x)}{2}} = \frac{2}{2-\cos(x)}$$

Please, show all work.