MATH 65B - Spring 2018

Groupwork 11: April 24, 2018

1. Determine whether the series converges or diverges.

$$(a) \qquad \sum_{n=1}^{\infty} \frac{1}{n(1+(\ln(n)^2)}$$

$$(b) \qquad \sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$$

<u>**Solution**</u>**:**

(a) We will apply the Integral Test for this question. Note that we have the condition $a_n = f(n)$ for $f(x) = \dfrac{1}{x(1+(\ln(x)^2))}$. We must show that $f(x)$ is continuous, positive, and decreasing on $[1,\infty)$. **Continuity**: $g(x) = x$ is a line therefore continuous, and $h(x) = (1+\ln(x)^2)$ is a composition of continuous function on $[1,\infty)$. Therefore the product $g(x)h(x)$ of continuous function is continuous, and the quotient $\frac{1}{g(x)h(x)}$ is also continuous as the denominator is non-zero on $[1,\infty)$. **Positive**: Since $x > 0$ on $[1,\infty)$, and $(1+\ln(x)^2)$ is also always positive on $[1,\infty)$, then $f(x)$ is positive on $[1,\infty)$. **Decreasing**: To show decreasing, we can show the derivative is negative:

$$f'(x) = -\frac{(1+\ln(x))^2}{x^2(1+(\ln(x))^2)^2} < 0,$$

where the inequality follows since all the terms in the fraction are positive, and the minus sign in front makes the whole function negative. Now we can apply the Integral Test:

$$\begin{aligned}
\int_1^\infty \frac{1}{x(1+(\ln(x)^2)}\,dx &= \lim_{t\to\infty}\int_1^t \frac{1}{x(1+(\ln(x)^2)}\,dx \\
&= \lim_{t\to\infty}\int_0^{\ln(t)} \frac{1}{1+u^2}\,du \qquad \text{let } u=\ln(x),\ du = \frac{1}{x}dx \\
&= \lim_{t\to\infty} \arctan(u)\big|_0^{\ln(t)} = \lim_{t\to\infty}\arctan(\ln(t)) - \arctan(0) \\
&= \frac{\pi}{2} < \infty \qquad \Rightarrow \quad \text{convergent by Integral Test}
\end{aligned}$$

The comparison or limit comparison test could also be used here to prove that the series is convergent.

(b) We will apply the Integral Test for this question, using the same format as the previous problem but for $f(x) = \dfrac{e^{1/x}}{x^2}$. **Continuity**: The functions $e^{1/x}$ and x^2, are both continuous, therefore their quotient is continuous on $[1,\infty)$, as x^2 is non-zero on $[1,\infty)$. **Positive**: Since the exponential function, and x^2 are always positive, $f(x)$ is also always positive on $[1,\infty)$. **Decreasing**: To show decreasing, we can show the derivative is negative:

$$f'(x) = -\frac{e^{1/x}(2x+1)}{x^4} < 0,$$

where the inequality follows since all the terms in the fraction are positive, and the minus sign in front makes the whole function negative. Now we can apply the Integral Test:

$$
\begin{aligned}
\int_1^\infty \frac{e^{1/x}}{x^2}\,dx &= \lim_{t\to\infty} \int_1^t \frac{e^{1/x}}{x^2}\,dx \\
&= \lim_{t\to\infty} -\int_1^{1/t} e^u\,du \qquad \text{let } u = \frac{1}{x},\ du = -\frac{1}{x^2}dx \\
&= \lim_{t\to\infty} e^u\big|_1^{1/t} = \lim_{t\to\infty} -e^{1/t} + e \\
&= e - 1 < \infty \qquad \Rightarrow \quad \text{convergent by Integral Test}
\end{aligned}
$$

The comparison or limit comparison test could also be used here to prove that the series is convergent.

Please, show all work.

2. Determine the values of p for which the series is convergent. You must show all work and justify your answer. *Hint: You can use the fact that* $f'(x) = -\frac{p+\ln(x)}{x^2(\ln(x))^{p+1}}$.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^p}$$

Solution:

We will apply the Integral Test for this question. Note that we have the condition $a_n = f(n)$ for $f(x) = \frac{1}{x(\ln(x))^p}$. We must show that $f(x)$ is continuous, positive, and decreasing on $[2, \infty)$. **Continuity**: $g(x) = x$ is a line therefore continuous, and $h(x) = (\ln(x))^p$ is also continuous for any p on $[2, \infty)$. So, the product and quotient of continuous functions is continuous, so $f(x)$ is continuous. **Positive**: Since $x > 0$ on $[2, \infty)$, and $(\ln(x))^p$ is also always positive on $[2, \infty)$ for any p, then $f(x)$ is positive on $[2, \infty)$. **Decreasing**: To show decreasing, we can show the derivative is negative:

$$f'(x) = -\frac{p+\ln(x)}{x^2(\ln(x))^{p+1}}.$$

Note that the denominator is always positive for any p, therefore we only need to work with the numerator. If we want $f'(x) < 0$, then we require that $p + \ln(x) > 0$, or solving for x, we have $x > e^{-p}$. So for all $x > e^{-p}$, the derivative will be negative, which is all we require to apply the Integral Test. Now we can apply the Integral Test:

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x(\ln(x))^p}\,dx &= \lim_{t\to\infty} \int_2^{t} \frac{1}{x(\ln(x))^p}\,dx \\
&= \lim_{t\to\infty} \int_{\ln(2)}^{\ln(t)} \frac{1}{u^p}\,du \qquad \text{let } u = \ln(x),\ du = \frac{1}{x}dx \\
&= \lim_{t\to\infty} \left.\frac{u^{-p+1}}{-p+1}\right|_{\ln(2)}^{\ln(t)} = \lim_{t\to\infty} \left.\frac{u^{1-p}}{1-p}\right|_{\ln(2)}^{\ln(t)} \qquad \text{assuming } p \neq 1 \\
&= \lim_{t\to\infty} \frac{\ln(t)^{1-p}}{1-p} - \frac{\ln(2)^{1-p}}{1-p} \\
&= \infty \quad \text{if } p < 1 \qquad \text{and } -\frac{\ln(2)^{1-p}}{1-p} < \infty \quad \text{if } p > 1
\end{aligned}$$

But what about $p = 1$? We can start with the second line above for the $p = 1$ case

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x\ln(x)}\,dx &= \lim_{t\to\infty} \int_{\ln(2)}^{\ln(t)} \frac{1}{u}\,du \\
&= \lim_{t\to\infty} \ln(u)\Big|_{\ln(2)}^{\ln(t)} \\
&= \lim_{t\to\infty} \ln(\ln(t)) - \ln(\ln(2)) \\
&= \infty \quad \Rightarrow \text{divergent for the case } p = 1
\end{aligned}$$

Therefore, the series is convergent for $p > 1$. (Recall: The $p = 2$ case was on the midterm!)

Please, show all work.

3. Determine whether the series converges or diverges.

$$(a) \qquad \sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2+1}$$

$$(b) \qquad \sum_{n=3}^{\infty} \frac{1}{\ln(\ln(n))}$$

Solution:

(a) We can apply the direct comparison test. Note that if we didn't have the $\cos^2(n)$ term, we would have just $\frac{1}{n^2+1}$. We also know that

$$n^2 < n^2+1 \qquad \text{for } n \geq 1$$
$$\frac{1}{n^2+1} < \frac{1}{n^2},$$

and $\frac{1}{n^2}$ is a convergent p-series. Thus we can use the above bound on our series to see that

$$\frac{\cos^2(n)}{n^2+1} \leq \frac{1}{n^2+1} < \frac{1}{n^2},$$

since $0 \leq \cos^2(x) \leq 1$. We now establish that

$$\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

So by direct comparison test, the series converges.

(b) We can apply the direct comparison test. To do so, we must remember our bound for the logarithm function

$$n > \ln(n) \qquad \text{for } n \geq 1$$
$$\ln(n) > \ln(\ln(n)) \qquad \text{for } n \geq 1$$
$$\Rightarrow \quad \ln(\ln(n)) < \ln(n) < n$$
$$\Rightarrow \quad \frac{1}{n} < \frac{1}{\ln(n)} < \frac{1}{\ln(\ln(n))}$$

where the second inequality follows since $\ln(x)$ is non-negative for all $x \geq 3$, and $\ln(x)$ is a strictly increasing function, so we can apply it to both sides of the inequality. In the third line, we can combine the inequalities, and in the fourth line, we use our usual reciprocal trick. Now, since the harmonic series diverges, we have

$$\infty = \sum_{n=3}^{\infty} \frac{1}{n} < \sum_{n=3}^{\infty} \frac{1}{\ln(n)} < \sum_{n=3}^{\infty} \frac{1}{\ln(\ln(n))},$$

so by direct comparison test, the series diverges.

Please, show all work.

4. Determine whether the series converges or diverges.

$$(a) \qquad \sum_{n=1}^{\infty} \frac{2n^5+4n^3+2n+1}{\sqrt{n^{12}+n^8+7n^2+1}}$$

$$(b) \qquad \sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

<u>**Solution:**</u>

(a) We should apply the limit comparison test due to the polynomials. The terms in the series are all positive, so we can apply limit comparison test. For these, take the highest powers of the numerator and the denominator, including any roots that are present to find out what to compare with

$$\frac{2n^5+4n^3+2n+1}{\sqrt{n^{12}+n^8+7n^2+1}} \qquad \Rightarrow \qquad \frac{n^5}{\sqrt{n^{12}}} = \frac{n^5}{n^6} = \frac{1}{n}.$$

So the sum of $\dfrac{1}{n}$ would give the harmonic series which is divergent. So we expect this series to be divergent. Take a_n to be our series and $b_n = \frac{1}{n}$. We now compute the limit and use some algebra

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2n^5+4n^3+2n+1}{\sqrt{n^{12}+n^8+7n^2+1}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{2n^6+4n^4+2n^2+n}{\sqrt{n^{12}+n^8+7n^2+1}}$$

$$= \lim_{n\to\infty} \frac{2n^6+4n^4+2n^2+n}{\sqrt{n^{12}+n^8+7n^2+1}} \cdot \frac{\frac{1}{\sqrt{n^{12}}}}{\frac{1}{\sqrt{n^{12}}}}$$

$$= \lim_{n\to\infty} \frac{2+\frac{4}{n^2}+\frac{2}{n^4}+\frac{1}{n^5}}{\sqrt{1+\frac{1}{n^4}+\frac{7}{n^{10}}+\frac{1}{n^{12}}}} = \frac{2}{\sqrt{1}} = 2$$

Since the limit is a finite positive number, our series is divergent since the harmonic series is divergent. **NOTE:** Just because the limit here is finite **does not imply** the series converges. Remember the theorem statement. So by limit comparison test, the series diverges.

(b) Use the Limit Comparison Test with the harmonic series. Note, we can use the limit comparison test since for $n = 1, 2, 3, \ldots$, we have that $0 \le \sin\left(\frac{1}{n}\right) \le \sin(1) \approx 0.841$, so all the terms are positive. We also use the trick from the previous quiz:

$$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{t\to 0^+} \frac{\sin(t)}{t} = 1$$

Therefore, since the limit is positive and finite, and the harmonic series diverges, the original series diverges by limit comparison test.

Please, show all work.