Last Name: First Name:

MATH 65B - Spring 2018

Groupwork 12: May 1, 2018

1. Use the Ratio Test to determine whether the series converges or diverges.

$$(a) \qquad \sum_{n=1}^{\infty} n! e^{-n}$$

$$(b) \qquad \sum_{n=1}^{\infty} \frac{n 2^n (n+1)!}{3^n n!}$$

<u>Solution</u>:

(a) Applying the Ratio Test, we see that

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(n+1)! e^{-(n+1)}}{n! e^{-n}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \cdot \frac{e^n}{e^{n+1}} \right| \\
&= \lim_{n\to\infty} \frac{(n+1)(n!)}{n!} \cdot \frac{e^n}{e^{n+1}} \\
&= \lim_{n\to\infty} \frac{n+1}{e} \\
&= \infty
\end{aligned}$$

So since the limit is equal to ∞, we have that the series is divergent by the Ratio Test.

(b) Applying the Ratio Test, we see that

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(n+1) 2^{n+1} (n+2)!}{3^{n+1} (n+1)!} \cdot \frac{3^n n!}{n 2^n (n+1)!} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+2)!}{(n+1)!} \cdot \frac{n!}{(n+1)!} \cdot \frac{(n+1)}{n} \cdot \frac{2^{n+1}}{2^n} \cdot \frac{3^n}{3^{n+1}} \right| \\
&= \lim_{n\to\infty} \frac{(n+2)(n+1)(2)}{(n+1)(n)(3)} \\
&= \frac{2}{3} \lim_{n\to\infty} \frac{n^2+3n+2}{n^2+n} \\
&= \frac{2}{3}(1) = \frac{2}{3}
\end{aligned}$$

So since the limit is equal to $\frac{2}{3} < 1$, we have that the series is convergent by the Ratio Test.

Please, show all work.

2. Use the Root Test to determine whether the series converges or diverges.

$$(a) \qquad \sum_{n=1}^{\infty} \frac{(-2)^{n-1}3^{n+1}}{n^n}$$

$$(b) \qquad \sum_{n=2}^{\infty} \frac{n}{(\ln(n))^{\frac{n}{2}}}$$

<u>Solution</u>:

(a) Applying the Root Test, we see that

$$\begin{aligned}
\lim_{n\to\infty} (|a_n|)^{\frac{1}{n}} &= \lim_{n\to\infty} \left(\left|\frac{(-2)^{n-1}3^{n+1}}{n^n}\right|\right)^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \left(\frac{2^{n-1}3^{n+1}}{n^n}\right)^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{(2)^{1-\frac{1}{n}}3^{1+\frac{1}{n}}}{n} \\
&= 0
\end{aligned}$$

So since the limit is equal to 0, we have that the series is convergent by the Root Test.

(b) Applying the Root Test, we see that

$$\begin{aligned}
\lim_{n\to\infty} (|a_n|)^{\frac{1}{n}} &= \lim_{n\to\infty} \left(\left|\frac{n}{(\ln(n))^{\frac{n}{2}}}\right|\right)^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \left(\frac{n}{(\ln(n))^{\frac{n}{2}}}\right)^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{n^{\frac{1}{n}}}{\sqrt{\ln(n)}} \\
&= \frac{\lim_{n\to\infty} n^{\frac{1}{n}}}{\lim_{n\to\infty} \sqrt{\ln(n)}} \\
&= \frac{\exp\left(\lim_{n\to\infty} \frac{\ln(n)}{n}\right)}{\lim_{n\to\infty} \sqrt{\ln(n)}} \\
&= \frac{\exp(0)}{\lim_{n\to\infty} \sqrt{\ln(n)}} \\
&= \frac{1}{\lim_{n\to\infty} \sqrt{\ln(n)}} \\
&= 0
\end{aligned}$$

So since the limit is equal to 0, we have that the series is convergent by the Root Test.

Please, show all work.

3. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$(a) \qquad \sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+1}$$

$$(b) \qquad \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^{\frac{3}{4}}}$$

<u>Solution</u>:

(a) First we note that the series is alternating. Before using the Alternating Series Test, we check absolute convergence:

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

which is divergent by the limit comparison test since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$$

So our original series is not absolutely convergent. To use the Alternating Series test, we must show the sequence has $\lim\limits_{n\to\infty} b_n = 0$, and the sequence b_n is decreasing. The limit is immediate as the highest power of the polynomial in the denominator is greater than the one in the numerator so the limit goes to zero as n approaches infinity. To show decreasing, either take the derivative or show directly. Using the theorem we look at

$$f(x) = \frac{x}{x^2+1} \quad \Rightarrow \quad f'(x) = \frac{1-x^2}{(x^2+1)^2} < 0 \quad \text{for } 1-x^2 < 0 \quad \text{or} \quad x > 1$$

thus we have that b_n is decreasing. So by Alternating Series Test we have convergence. But note that since the original series is not absolutely convergent, this series is **conditionally convergent**.

(b) First note that $\cos(n\pi) = (-1)^n$ for $n = 1, 2, 3, \ldots$, so the series is alternating. First we check absolute convergence:

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{4}}}$$

which is a divergent p-series, so the original series is not absolutely convergent. To use the Alternating Series test, we must show the sequence has $\lim\limits_{n\to\infty} b_n = 0$, and the sequence b_n is decreasing. The limit is immediate as the denominator approaches infinity as n approaches infinity, so the overall limit goes to zero. To show decreasing, either take the derivative or show directly. Using the theorem we look at

$$f(x) = \frac{1}{x^{\frac{3}{4}}} \quad \Rightarrow \quad f'(x) = -\frac{3}{4x^{\frac{7}{4}}} < 0 \quad \text{for } x \geq 1$$

thus we have that b_n is decreasing. So by Alternating Series Test we have convergence. But note that since the original series is not absolutely convergent, this series is **conditionally convergent**.

Please, show all work.

4. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$(a) \qquad \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{\ln(n)}$$

$$(b) \qquad \sum_{n=1}^{\infty} \frac{(-1)^n \arctan(n)}{n^2}$$

__Solution:__

(a) First we note that the series is alternating. Before using the Alternating Series Test, we check absolute convergence:

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{\ln(n)}$$

which is divergent by the limit comparison test with the harmonic series. So our original series is not absolutely convergent. To use the Alternating Series test, we must show the sequence has $\lim_{n\to\infty} b_n = 0$, and the sequence b_n is decreasing. The limit is immediate as $\ln(n)$ approaches infinity as n approaches infinity, and it is in the denominator, so the overall limit goes to 0. To show decreasing, either take the derivative or show directly. Using the theorem we look at

$$\begin{aligned} \ln(n) &< \ln(n+1) && \text{since } \ln(x) \text{ is a strictly increasing function} \\ \frac{1}{\ln(n+1)} &< \frac{1}{\ln(n)} && \Rightarrow b_{n+1} < b_n \end{aligned}$$

thus we have that b_n is decreasing. So by Alternating Series Test we have convergence. But note that since the original series is not absolutely convergent, this series is **conditionally convergent**.

(b) First we note that the series is alternating. Before using the Alternating Series Test, we check absolute convergence:

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{\arctan(n)}{n^2} \leq \sum_{n=1}^{\infty} \frac{\frac{\pi}{2}}{n^2} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

and the series of $\frac{1}{n^2}$ is a convergent p-series, so the original series is absolutely convergent by the comparison test.