LAST NAME: FIRST NAME:

MATH 65B - Spring 2018

Groupwork 8: March 29, 2018

1. Give a sketch of the following polar curves. Be sure to label the x and y intercepts of the curves.

(a) $r = \sin(\theta)$

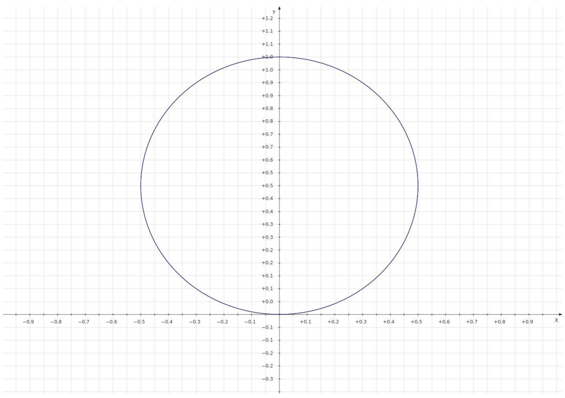
(b) $r = 4\sin(3\theta)$

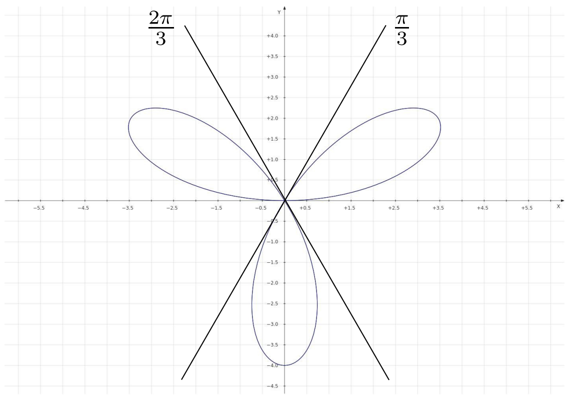
Solution:

(a) Note that we can eliminate the parameter by using polar coordinate formulas:

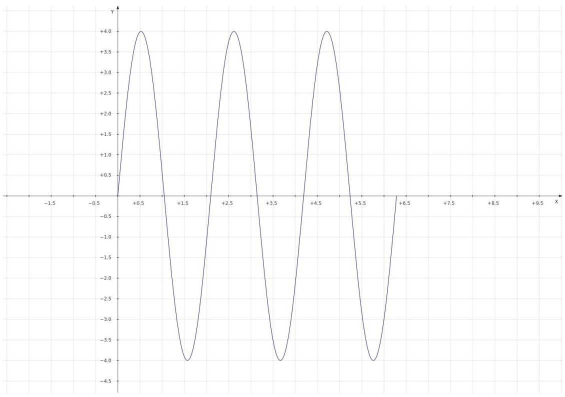
$$\begin{aligned} r = \sin(\theta) \;&\Rightarrow\; r^2 = r\sin(\theta) \;\Rightarrow\; x^2 + y^2 = y \\ &\Rightarrow\; x^2 + y^2 - y = 0 \;\Rightarrow\; x^2 + y^2 - y + \frac{1}{4} = \frac{1}{4} \\ &\Rightarrow\; x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2 \end{aligned}$$

The graph is a circle of radius $\frac{1}{2}$, with center at $(0, \frac{1}{2})$. It is graphed in the figure on the left below.





(b) For $r = 4\sin(3\theta)$, it is helpful to graph this function in the $r\theta$-parameter plane. The



plot of $r = 4\sin(3\theta)$ for $0 \le \theta \le 2\pi$, in the $r\theta$-parameter plane is given directly above, and shows what the radius is for a given θ. Notice that the $r = 0$ points occur at $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$, and 2π. These are the points that correspond to the lines of the graph in xy-space. Notice also that halfway between the $r = 0$ points, we have the maximum and minimums at $r = \pm 4$. These are the tips of the petals in the plot.

Please, show all work.

2. (a) Find the slope of the tangent line to the given polar curve at the point specified by the value of θ.

$$r = 2\sin(\theta), \qquad \theta = \frac{\pi}{6}$$

(b) Find the values of θ for the given polar curve where the tangent line is horizontal and vertical (Restrict to $0 \leq \theta \leq 2\pi$).

$$r = 2\sin(\theta)$$

<u>Solution</u>:

(a) By direct computation:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)} = \frac{2\cos(\theta)\sin(\theta) + 2\sin(\theta)\cos(\theta)}{2\cos^2(\theta) - 2\sin^2(\theta)} \\
&= \frac{4\cos(\theta)\sin(\theta)}{2(\cos^2(\theta) - \sin^2(\theta))} = \frac{2\sin(2\theta)}{2(\cos^2(\theta) - \sin^2(\theta))} \\
&= \frac{2\frac{\sqrt{3}}{2}}{2\left(\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right)} \\
&= \sqrt{3}
\end{aligned}$$

(b) From part (a), we already have the derivative $\dfrac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{2\sin(2\theta)}{2(\cos^2(\theta) - \sin^2(\theta))}$$

First we will find where the numerator and denominator are equal to zero, then consider the various situations.

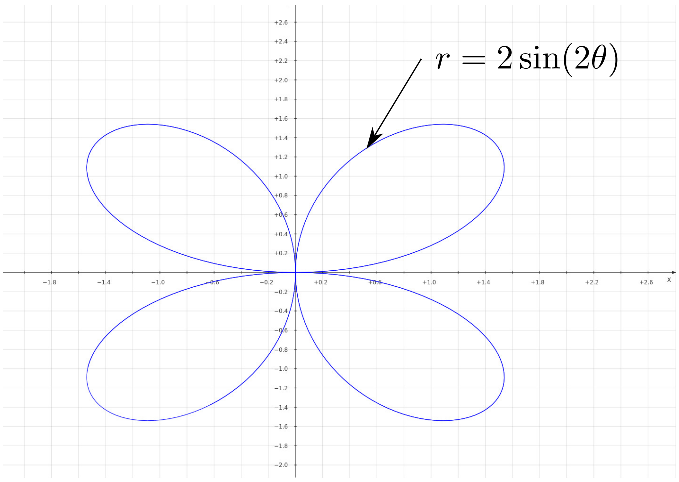
$$\begin{aligned}
\frac{dy}{d\theta} &= 0 = 2\sin(2\theta) \\
&\Rightarrow\ 0 = \sin(2\theta) \\
&\Rightarrow\ 0 = \sin(u) \qquad \text{for } u = 2\theta \\
&\Rightarrow\ u = 0, \pi, 2\pi, 3\pi, 4\pi \\
&\Rightarrow\ \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi \\
\frac{dx}{d\theta} &= 0 = 2(\cos^2(\theta) - \sin^2(\theta)) \\
&\Rightarrow\ 0 = \cos^2(\theta) - \sin^2(\theta) \\
&\Rightarrow\ 0 = \cos(2\theta) \qquad \text{using identity } \cos^2(\theta) - \sin^2(\theta) = \cos(2\theta) \\
&\Rightarrow\ 0 = \cos(u) \qquad \text{for } u = 2\theta \\
&\Rightarrow\ u = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \\
&\Rightarrow\ \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}
\end{aligned}$$

Since none of the values appear in both $\dfrac{dy}{d\theta} = 0$ and $\dfrac{dx}{d\theta} = 0$, we don't have to worry about $\dfrac{0}{0}$. Therefore, we have horizontal tangents at $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$, and vertical tangents at $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

Please, show all work.

3. Find the area of one petal of the polar rose given by:

$$r = 2\sin(2\theta)$$



<u>Solution</u>:

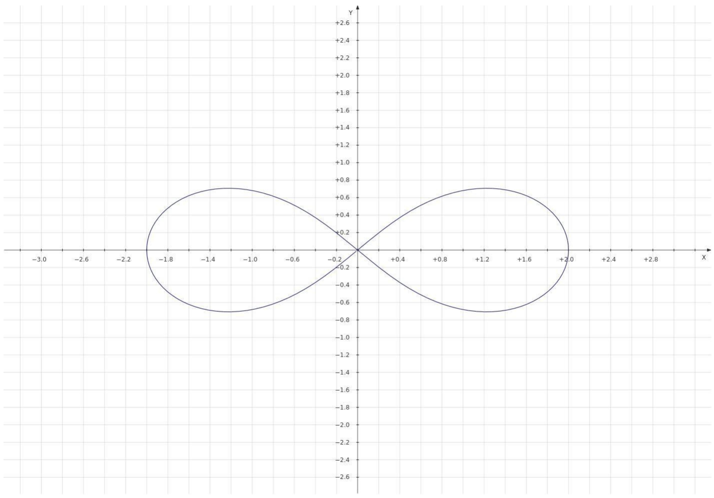
Note that the curve starts at the origin (0,0), and then at $\theta = \pi/4$, the radius is equal to 1, so the coordinate is $(1, \pi/4)$ in the first quadrant. Then, for $\theta = \pi/2$, the graph is back to the origin (0,0). So from $\theta = 0$ to $\theta = \pi/2$, the petal in the first quadrant is traced out. Now we can compute the area of the first petal:

$$\begin{aligned}
\int_a^b \frac{1}{2}(f(\theta))^2 \, d\theta &= \int_0^{\frac{\pi}{2}} \frac{1}{2}(2\sin(2\theta))^2 \, d\theta \\
&= \int_0^{\frac{\pi}{2}} 2\sin^2(2\theta) \, d\theta \\
&= \int_0^{\frac{\pi}{2}} (1 - \cos(2\theta)) \, d\theta \\
&= \left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\pi/2} \\
&= \frac{\pi}{2}
\end{aligned}$$

Please, show all work.

4. Find the area of region bounded by the lemniscate.

$$r^2 = 4\cos(2\theta)$$



Solution:

We will plot the curve in 4 steps. Note that at $\theta = 0$, we get $r^2 = 4$, so the curve starts at the origin $(r, \theta) = (2, 0)$ or $(x, y) = (2, 0)$, and then at $\theta = \pi/4$, we have $r^2 = 0$, so the coordinate is $(r, \theta) = (0, \pi/4)$ or $(x, y) = (0, 0)$. Notice that in the interval $[\pi/4, \pi/2]$ the value of $\cos(2\theta)$ is negative, so there is no real value of r that satisfies this equation. Continuing to plug in values, we get the following chart.

θ	r^2	r
0	4	± 2
$\frac{\pi}{6}$	2	$\pm\sqrt{2}$
$\frac{\pi}{4}$	0	0
$\frac{\pi}{3}$	-2	XXX
$\frac{\pi}{2}$	-4	XXX

θ	r^2	r
$\frac{2\pi}{3}$	-2	XXX
$\frac{3\pi}{4}$	0	0
$\frac{5\pi}{6}$	2	$\pm\sqrt{2}$
π	4	$\pm$ 2

Notice that we can trace out a half petal over the region $0 \leq \theta \leq \frac{\pi}{4}$. So we can thus compute the area of that half petal, then by symmetry, multiply the result by 4 to get the full area. Note that we have $r^2 = 4\cos(2\theta)$, so we don't need to square the function.

$$\begin{aligned} \text{Area of all petals} = 4\int_a^b \frac{1}{2}(f(\theta))^2\, d\theta &= 4\int_0^{\frac{\pi}{4}} \frac{1}{2} \cdot 4\cos(2\theta)\, d\theta \\ &= 8\int_0^{\frac{\pi}{4}} \cos(2\theta)\, d\theta \\ &= 8\left(\frac{1}{2}\sin(2\theta)\right)\Big|_0^{\frac{\pi}{4}} \\ &= 4\sin\left(\frac{\pi}{2}\right) \\ &= 4 \end{aligned}$$

Please, show all work.