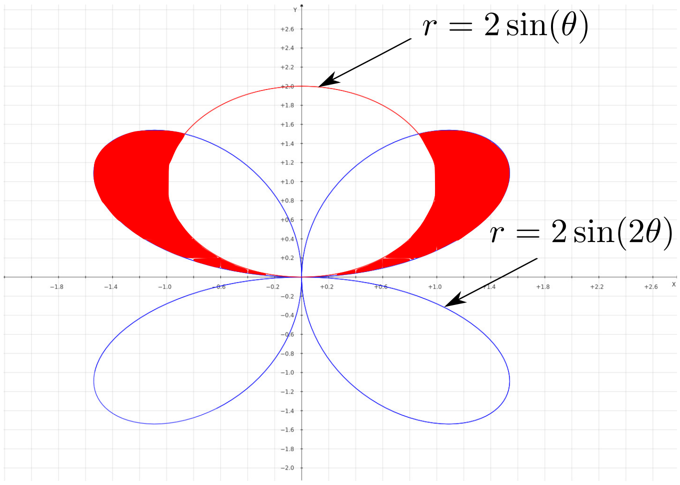
Last Name: First Name:

MATH 65B - Spring 2018

Groupwork 9: April 5, 2018

1. Find the area of the region that lies outside the circle $r = 2\sin(\theta)$, and inside the polar rose $r = 2\sin(2\theta)$. The region is given by the shaded region in the labeled plot below.



Solution:

Find the intersection points first! We solve the following:

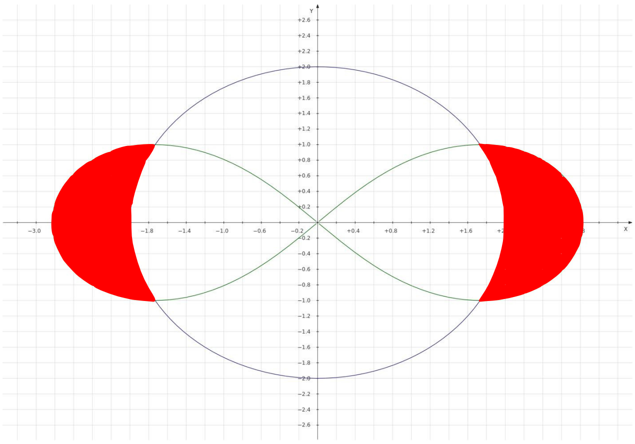
$$\begin{aligned} 2\sin(\theta) &= 2\sin(2\theta) \\ 2\sin(\theta) &= 4\sin(\theta)\cos(\theta) \\ 2\sin(\theta)\cos(\theta) - \sin(\theta) &= 0 \\ \sin(\theta)(2\cos(\theta) - 1) &= 0 \\ \sin(\theta) = 0 \quad \text{and} \quad \cos(\theta) &= \frac{1}{2} \\ \Rightarrow \quad \theta &= 0, \pi, \frac{\pi}{3}, \frac{5\pi}{3} \end{aligned}$$

Now, we can find the area of the shaded region on the left, and multiply by 2 due to symmetry. The "outside" region is the polar rose, and the "inside" region is the circle. So we integrate only to the first intersection point, ie. 0 to $\frac{\pi}{3}$. So we have

$$\begin{aligned} A &= 2\int_a^b \frac{1}{2}\left(f(\theta)^2 - g(\theta)^2\right) d\theta = 2\int_0^{\frac{\pi}{3}} \frac{1}{2}\left((2\sin(2\theta))^2 - (2\sin(\theta))^2\right) d\theta \\ &= 4\int_0^{\frac{\pi}{3}} \left(\sin^2(2\theta) - \sin^2(\theta)\right) d\theta \\ &= 4\int_0^{\frac{\pi}{3}} \left(\left(\frac{1}{2}(1 - \cos(4\theta))\right) - \left(\frac{1}{2}(1 - \cos(2\theta))\right)\right) d\theta \\ &= 2\int_0^{\frac{\pi}{3}} (\cos(2\theta) - \cos(4\theta))\, d\theta \\ &= 2\left[\frac{1}{2}\sin(2\theta) - \frac{1}{4}\sin(4\theta)\right]_0^{\frac{\pi}{3}} = \frac{3\sqrt{3}}{4} \end{aligned}$$

Please, show all work.

2. Find the area inside the lemniscate $r^2 = 8\cos(2\theta)$, and outside the circle $r = 2$.



Solution:

Find the intersection points first! We solve the following:

$$\begin{aligned}
r^2 = 8\cos(2\theta) \quad &\text{and} \quad r^2 = 4 \\
8\cos(2\theta) &= 4 \\
\cos(2\theta) &= \frac{1}{2} \\
\cos(u) &= \frac{1}{2} \qquad \text{for } u = 2\theta \\
u &= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3} \\
\theta &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}
\end{aligned}$$

Now, we can find the area of the shaded region on the left, and multiply by 4 due to symmetry. The "outside" region is the lemniscate half petal, and the "inside" region is the circle. So we integrate only to the first intersection point, ie. 0 to $\frac{\pi}{6}$. So we have

$$\begin{aligned}
A &= 4\int_a^b \frac{1}{2}\left(f(\theta)^2 - g(\theta)^2\right) d\theta \\
&= 4\int_0^{\frac{\pi}{6}} \frac{1}{2}\left(8\cos(2\theta) - (2)^2\right) d\theta \\
&= 8\int_0^{\frac{\pi}{6}} \left(2\cos(2\theta) - 1\right) d\theta \\
&= 8\left[\sin(2\theta) - \theta\right]\Big|_0^{\frac{\pi}{6}} \\
&= 4\sqrt{3} - \frac{4\pi}{3}
\end{aligned}$$

Please, show all work.

3. Find the length of the given curve.

$$r = \cos^4\left(\frac{\theta}{4}\right), \quad 0 < \theta < 2\pi$$

__Solution__**:**

Recall that the arc length formula for polar coordinates is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

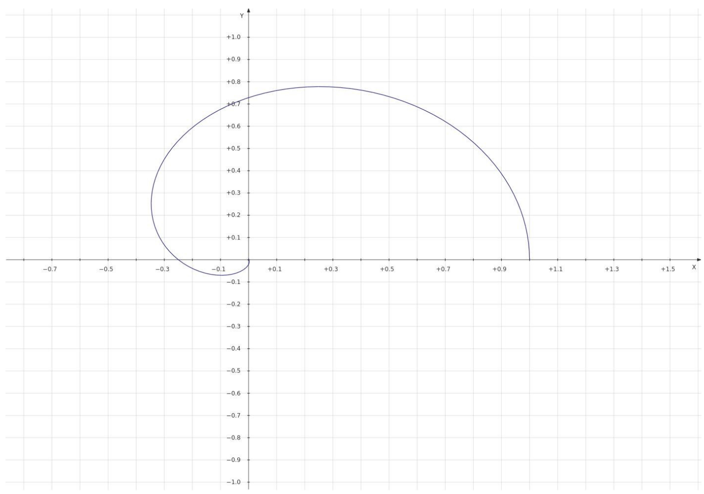
First we compute the inside of the square root

$$\begin{aligned} r^2 + \left(\frac{dr}{d\theta}\right)^2 &= \cos^8\left(\frac{\theta}{4}\right) + \left(4\cos^3\left(\frac{\theta}{4}\right)\left(-\sin\left(\frac{\theta}{4}\right)\right)\cdot\frac{1}{4}\right)^2 \\ &= \cos^8\left(\frac{\theta}{4}\right) + \cos^6\left(\frac{\theta}{4}\right)\sin^2\left(\frac{\theta}{4}\right) \\ &= \cos^6\left(\frac{\theta}{4}\right)\left[\cos^2\left(\frac{\theta}{4}\right) + \sin^2\left(\frac{\theta}{4}\right)\right] \\ &= \cos^6\left(\frac{\theta}{4}\right) \end{aligned}$$

Now we can compute the integral:

$$\begin{aligned} L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{\cos^6\left(\frac{\theta}{4}\right)}\, d\theta &= \int_0^{2\pi} \left|\cos^3\left(\frac{\theta}{4}\right)\right|\, d\theta \\ &= \int_0^{2\pi} \cos^3\left(\frac{\theta}{4}\right)\, d\theta \qquad \text{note that } \cos^3\left(\frac{\theta}{4}\right) \geq 0 \text{ for } 0 \leq \theta \leq 2\pi \\ &= 4\int_0^{\frac{\pi}{2}} \cos^3(u)\, du \qquad \text{use } u = \frac{\theta}{4},\ du = \frac{1}{4}d\theta \\ &= 4\left[\sin(u) - \frac{\sin^3(u)}{3}\right]\Bigg|_0^{\frac{\pi}{2}} = 4\left(1 - \frac{1}{3}\right) = \frac{8}{3} \end{aligned}$$

The plot of $r = \cos^4\left(\frac{\theta}{4}\right), \quad 0 < \theta < 2\pi$ is given below



Please, show all work.