

These notes cover examples from the lecture on § 7.1 - Inverse Sine, Cosine, and Tangent Functions, as well as some extra examples. These cover the most important types you are likely to see. It is **strongly recommended** that you work more problems similar to these in order to get good at these types of problems as they are very likely to show up on quizzes and tests.

1. § 7.1 - INVERSE SINE, COSINE, AND TANGENT FUNCTIONS

Below is summary of the content for this section. Specifically, the range of the inverse trig functions is important in determining the solutions to the problems. This chart should be used as a quick reference guide. A detailed explanation is given in each subsection. Note that we are restricting the range of some of the trig functions listed below.

Function	Domain	Range
$\sin(x)$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$[-1, 1]$
$\cos(x)$	$[0, \pi]$	$[-1, 1]$
$\tan(x)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$(-\infty, \infty)$
$\sec(x)$	$[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$	$(-\infty, -1] \cup [1, \infty)$
$\csc(x)$	$[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$	$(-\infty, -1] \cup [1, \infty)$
$\cot(x)$	$(0, \pi)$	$(-\infty, \infty)$
$\sin^{-1}(x)$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\cos^{-1}(x)$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}(x)$	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$
$\sec^{-1}(x)$	$(-\infty, -1] \cup [1, \infty)$	$[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$
$\csc^{-1}(x)$	$(-\infty, -1] \cup [1, \infty)$	$[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$
$\cot^{-1}(x)$	$(-\infty, \infty)$	$(0, \pi)$

Definition 1.1. We define *inverse sine* function to be

$$y = \sin^{-1}(x) \text{ means } x = \sin(y)$$

$$\text{for } -1 \leq x \leq 1 \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

What $y = \sin^{-1}(x)$ means is that “ y is the angle whose sine is equal to x .” Or in other words, what angle y gives you the value of x . **NOTE:** $y = \sin^{-1}(x)$ does **NOT** mean $y = \frac{1}{\sin(x)}$! We know that this function is cosecant. The -1 is purely notation, if you like, you can write the expression as $y = \sin^{-1}(x) = \arcsin(x)$, which is another name for the inverse sine function.

Using the notion of “what angle y gives you the value x ” in the formula $y = \sin^{-1}(x)$, we can figure out the values from the table below of $\sin(\theta)$:

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin(\theta)$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Note that for $f^{-1}(x) = \sin^{-1}(x)$, the domain is $[-1, 1]$ and the range is $[-\frac{\pi}{2}, \frac{\pi}{2}]$, reverse of $\sin(x)$. Try to do the following exercises without the chart, using the method: “what angle y gives you the value x .”

Example 1. Find the exact value of $\sin^{-1}(-\frac{1}{2})$.

Example 2. Find the exact value of $\sin^{-1}(-1)$.

Example 3. Find the exact value of $\sin^{-1}(\frac{\sqrt{3}}{2})$.

Example 4. Find the exact value of $\sin^{-1}(\frac{\sqrt{2}}{2})$.

Sometimes we can use the fact that $\sin(x)$ and $\sin^{-1}(x)$ are inverses directly, but you **MUST** be careful! The values need to be in the domain and range of \sin^{-1} to be well defined to be able to apply the following rules.

Definition 1.2. For values x in the domain and range of \sin^{-1} to cancel the \sin and \sin^{-1} :

$$f^{-1}(f(x)) = \sin^{-1}(\sin(x)) = x \text{ for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$f(f^{-1}(x)) = \sin(\sin^{-1}(x)) = x \text{ for } -1 \leq x \leq 1$$

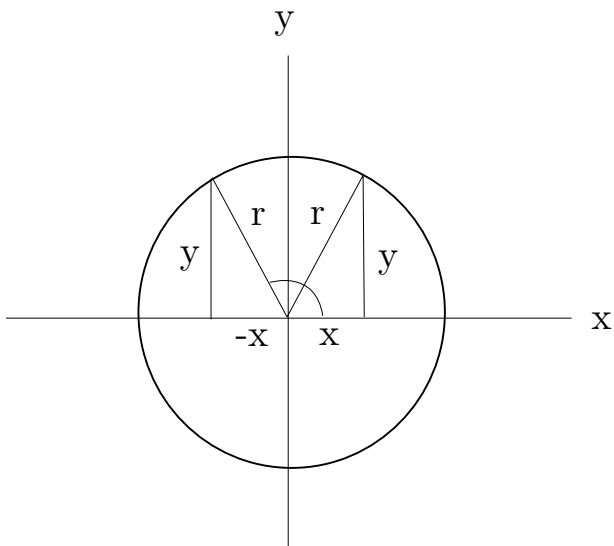


FIGURE 1. Reference angle for Example 6

Here are some examples where you may or may not be able to use this definition. Always check to see if the values are in the above intervals!

Example 5. Find the exact value of $\sin^{-1}(\sin(\frac{\pi}{8}))$.

Example 6. Find the exact value of $\sin^{-1}(\sin(\frac{5\pi}{8}))$.

Solution: For this question we have to check to see if the angle is $\frac{5\pi}{8}$ in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. This angle is **NOT** in the interval. So, we need to find the correct angle in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ that gives the same value as $\sin(\frac{5\pi}{8})$. Since we are dealing with $\sin(\theta)$, we have the formula $\sin(\theta) = \frac{y}{r}$. The angle $\frac{5\pi}{8}$ is in quadrant II, this is the angle that has the arc in the figure above. The only other possible triangle that gives us the same value $\sin(\theta) = \frac{y}{r}$. The smaller angle (what we want to find, is the reference angle) is found by doing

$$\theta = \pi - \frac{5\pi}{8} = \frac{3\pi}{8}$$

since going from the positive x-axis to the negative x-axis covers π radians and the measured angle is $\frac{5\pi}{8}$, so we get their difference. Therefore, our solution is

$$\sin^{-1}(\sin(\frac{5\pi}{8})) = \frac{3\pi}{8}$$

□

Example 7. Find the exact value of $\sin(\sin^{-1}(\frac{1}{2}))$.

Example 8. Find the exact value of $\sin(\sin^{-1}(1.8))$.

Definition 1.3. We define *inverse cosine* function to be

$$y = \cos^{-1}(x) \text{ means } x = \cos(y) \\ \text{for } -1 \leq x \leq 1 \text{ and } 0 \leq y \leq \pi$$

What $y = \cos^{-1}(x)$ means is that “ y is the angle whose cosine is equal to x .” Or in other words, what angle y gives you the value of x . **NOTE:** $y = \cos^{-1}(x)$ does **NOT** mean $y = \frac{1}{\cos(x)}$! We know that this function is secant. The -1 is purely notation, if you like, you can write the expression as $y = \cos^{-1}(x) = \arccos(x)$, which is another name for the inverse cosine function.

Using the notion of “what angle y gives you the value x ” in the formula $y = \cos^{-1}(x)$, we can figure out the values from the table below of $\cos(\theta)$:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

Note that for $\sin^{-1}(x)$, the domain is $[-1, 1]$ and the range is $[0, \pi]$. Try to do the following exercises without the chart, using the method: “what angle y gives you the value x .”

Example 9. Find the exact value of $\cos^{-1}(-\frac{1}{2})$.

Example 10. Find the exact value of $\cos^{-1}(0)$.

Example 11. Find the exact value of $\cos^{-1}(\frac{\sqrt{3}}{2})$.

Example 12. Find the exact value of $\cos^{-1}(\frac{\sqrt{2}}{2})$.

Sometimes we can use the fact that $\cos(x)$ and $\cos^{-1}(x)$ are inverses directly, but you **MUST** be careful! The values need to be in the domain and range of \cos^{-1} to be well defined to be able to apply the following rules.

Definition 1.4. For values x in the domain and range of \cos^{-1} to cancel the \cos and \cos^{-1} :

$$f^{-1}(f(x)) = \cos^{-1}(\cos(x)) = x \text{ for } 0 \leq x \leq \pi \\ f(f^{-1}(x)) = \cos(\cos^{-1}(x)) = x \text{ for } -1 \leq x \leq 1$$

Here are some examples where you may or may not be able to use this definition. Always check to see if the values are in the above intervals!

Example 13. Find the exact value of $\cos^{-1}(\cos(\frac{\pi}{12}))$.

Example 14. Find the exact value of $\cos(\cos^{-1}(-0.4))$.

Example 15. Find the exact value of $\cos^{-1}(\cos(-\frac{2\pi}{3}))$.

Example 16. Find the exact value of $\cos(\cos^{-1}(\pi))$.

Definition 1.5. We define *inverse tangent* function to be

$$y = \tan^{-1}(x) \text{ means } x = \tan(y) \\ \text{for } -1 \leq x \leq 1 \text{ and } 0 \leq y \leq \pi$$

What $y = \tan^{-1}(x)$ means is that “ y is the angle whose tangent is equal to x .” Or in other words, what angle y gives you the value of x . **NOTE:** $y = \tan^{-1}(x)$ does **NOT** mean $y = \frac{1}{\tan(x)}$! We know that this function is cotangent. The -1 is purely notation, if you like, you can write the expression as $y = \tan^{-1}(x) = \arctan(x)$, which is another name for the inverse tangent function.

Using the notion of “what angle y gives you the value x ” in the formula $y = \tan^{-1}(x)$, we can figure out the values from the table below of $\tan(\theta)$:

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan(\theta)$	undefined	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

Note that the domain is $(-\infty, \infty)$ and the range is $(-\frac{\pi}{2}, \frac{\pi}{2})$. Try to do the following exercises without the chart, using the method: “what angle y gives you the value x .”

Example 17. Find the exact value of $\tan^{-1}(-\sqrt{3})$.

Example 18. Find the exact value of $\tan^{-1}(1)$.

Example 19. Find the exact value of $\tan^{-1}(\frac{\sqrt{3}}{3})$.

Example 20. Find the exact value of $\tan^{-1}(0)$.

Sometimes we can use the fact that $\tan(x)$ and $\tan^{-1}(x)$ are inverses directly, but you **MUST** be careful! The values need to be in the domain and range of \tan^{-1} to be well defined to be able to apply the following rules.

Definition 1.6. For values x in the domain and range of \tan^{-1} to cancel the \tan and \tan^{-1} :

$$f^{-1}(f(x)) = \tan^{-1}(\tan(x)) = x \text{ for } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ f(f^{-1}(x)) = \tan(\tan^{-1}(x)) = x \text{ for } -\infty < x < \infty$$

Here are some examples where you may or may not be able to use this definition. Always check to see if the values are in the above intervals!

Example 21. Find the exact value of $\tan^{-1}(\tan(-\frac{3\pi}{8}))$.

Example 22. Find the exact value of $\tan(\tan^{-1}(4))$.

Example 23. Find the exact value of $\tan^{-1}(\tan(-\frac{2\pi}{3}))$.

Example 24. Find the exact value of $\tan(\tan^{-1}(\pi))$.

Here is a chart to summarize the above information. This is how to use the chart. When encountering problems of the form:

$$\begin{aligned} \text{(A)} \quad & \sin^{-1}(\sin(\theta)) = \phi \\ \text{(B)} \quad & \sin(\sin^{-1}(x)) = y \end{aligned}$$

Look at the function on the outside, then reference the table. For the **(A)**, look at the first three rows, if θ is in the interval in the second column, then you can just cancel the trig functions and $\phi = \theta$. For the **(B)**, look at the last three rows, if x is in the interval in the second column, then you can just cancel the trig functions and $y = x$. For case **(A)**, if the value θ is not in the interval, you must do more work and find the correct angle. The answer will **NOT** be the one in parentheses. For case **(B)**, if the value x is not in the interval, the value will not be defined. The answer will **NOT** be the one in parentheses.

Trig Function on the outside	Interval
\sin^{-1}	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
\cos^{-1}	$[0, \pi]$
\tan^{-1}	$(-\frac{\pi}{2}, \frac{\pi}{2})$
\sin	$[-1, 1]$
\cos	$[-1, 1]$
\tan	$(-\infty, \infty)$

Here I will do one worked example of each of the two last topics. The first is finding the inverse function of a trigonometric function. The second is solving an equation involving an inverse trigonometric function.

Example 25. Find the inverse function f^{-1} of $f(x) = 2\sin(x) - 1$ on $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Find the range of f and the domain and range of f^{-1} .

Solution: Using our previous knowledge, we switch x and y and solve for y .

$$\begin{aligned} y &= 2\sin(x) - 1 \\ x &= 2\sin(y) - 1 \\ x + 1 &= 2\sin(y) \\ \frac{x + 1}{2} &= \sin(y) \\ y &= \sin^{-1}\left(\frac{x + 1}{2}\right) \end{aligned}$$

So the inverse function is $f^{-1}(x) = \sin^{-1}\left(\frac{x+1}{2}\right)$. Now we have to find the range of f . To do this, we solve $y = 2\sin(x) - 1$ for $\sin(x)$. So we have that $\sin(x) = \frac{y+1}{2}$. We know that $-1 \leq \sin(x) \leq 1$, so then we must have the following

$$\begin{aligned} -1 &\leq \frac{y+1}{2} \leq 1 \\ -2 &\leq y+1 \leq 2 \\ -3 &\leq y \leq 1 \end{aligned}$$

So in interval notation, the range is $[-3, 1]$. Therefore, for f^{-1} , we switch the domain and the range. So then we have that the domain for f^{-1} is $[-3, 1]$ and the range is $[-\frac{\pi}{2}, \frac{\pi}{2}]$. \square

Example 26. Solve the equation: $3\sin^{-1}(x) = \pi$.

Solution: To solve this type of question, isolate the trig function and use the definition from above:

$$\begin{aligned} 3 \sin^{-1}(x) &= \pi \\ \sin^{-1}(x) &= \frac{\pi}{3} \\ x &= \sin\left(\frac{\pi}{3}\right) \\ x &= \frac{\sqrt{3}}{2} \end{aligned}$$

□

Here are some examples to practice of these two ideas.

Example 27. Find the inverse function f^{-1} of $f(x) = 5 \sin(x) + 2$ on $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Find the range of f and the domain and range of f^{-1} .

Example 28. Find the inverse function f^{-1} of $f(x) = -2 \cos(x)$ on $0 \leq x \leq \frac{\pi}{3}$. Find the range of f and the domain and range of f^{-1} .

Example 29. Find the inverse function f^{-1} of $f(x) = 3 \sin(2x)$ on $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$. Find the range of f and the domain and range of f^{-1} .

Example 30. Solve the equation: $-6 \sin^{-1}(x) = \pi$.

Example 31. Solve the equation: $3 \tan^{-1}(x) = \pi$.

Example 32. Solve the equation: $3 \cos^{-1}(2x) = 2\pi$.

Inverse Trigonometric Functions

- The graphs of the six inverse trigonometric functions are shown here.

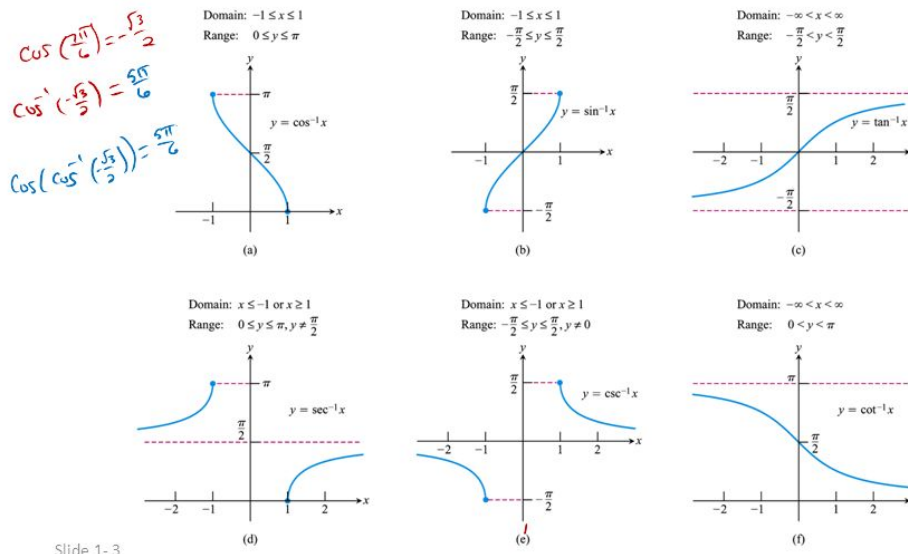


FIGURE 2. Graphs of Inverse Trig Functions. Picture from <http://images.slideplayer.com/27/8963056/slides>