Section 8.5 - Partial Fractions

Recall how to write a polynomial expression as partial fractions

$$\frac{x+5}{x^2+x-2} = \frac{x+5}{(x-1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)} = \frac{A(x+2)+B(x-1)}{(x+2)(x-1)}$$

So $x+5 = A(x+2)+B(x-1)$

$x+5 = (A+B)x + (2A-B) \Rightarrow \begin{matrix} A+B=1 \\ 2A-B=5 \end{matrix} \Rightarrow \boxed{\begin{matrix} A=2 \\ B=-1 \end{matrix}}$

or

$x+5 = A(x+2)+B(x-1)$ Let $x=1 \Rightarrow 1+5=3A$

$\Rightarrow \boxed{A=2}$

$x+5 = A(x+2)+B(x-1)$ Let $x=-2 \Rightarrow -2+5=-3B$

$\Rightarrow \boxed{B=-1}$

$$\Rightarrow \frac{x+5}{x^2+x-2} = \frac{2}{x-1} + \frac{-1}{x+2}$$

Suppose we want to integrate: $\int \frac{x+5}{x^2+x-2}\,dx$

$$\Rightarrow \int \frac{x+5}{x^2+x-2}\,dx = \int \frac{2}{x-1}\,dx + \int \frac{-1}{x+2}\,dx$$

$$= \boxed{2\ln|x-1| - \ln|x+2| + C}$$

Ex) Sometimes we can polynomial divide: $\int \frac{x^3+x}{x-1}\,dx$

$$\begin{array}{r} x^2+x+2 \\ x-1 \overline{) x^3 + x} \\ -(x^3-x^2) \\ x^2+x \\ -(x^2-x) \\ 2x \\ -(2x-2) \\ 2 \end{array}$$

$$\Rightarrow \int \frac{x^3+x}{x-1}\,dx = \int \left(x^2+x+2+\frac{2}{x-1}\right)dx$$

$$= \boxed{\frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C}$$

So we are looking at cases of $f(x) = \frac{P(x)}{Q(x)}$ being the integrand. We have 4 cases

Case I – $Q(x)$ is a product of distinct linear terms

Case II – $Q(x)$ is a product of linear factors, some repeats

Case III – $Q(x)$ has irreducible quadratic terms, no repeats

Case IV – $Q(x)$ has repeated irreducible quadratic factor

Ex) (Case I) $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}$ $\qquad 2x^3+3x^2-2x = x(2x-1)(x+2)$

$$\Rightarrow \frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

$$\Rightarrow x^2+2x-1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

$$\Rightarrow A = \tfrac{1}{2},\ B = \tfrac{1}{5},\ C = -\tfrac{1}{10}$$

$$\Rightarrow = \frac{1}{2}\int \frac{1}{x}\,dx + \frac{1}{5}\int \frac{1}{2x-1}\,dx - \frac{1}{10}\int \frac{1}{x+2}\,dx$$

$$= \boxed{\frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C}$$

Ex) (Case 2) $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$

Long division $\Rightarrow$ $\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1 + \frac{4x}{x^3-x^2-x+1}$

$$= x+1 + \frac{4x}{(x-1)^2(x+1)}$$

$$\Rightarrow \frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

[Side Note: So if $\frac{x^3-x+1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$

We put all powers from 1 up to degree n of the repeated factor]

$$\Rightarrow 4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

$$\Rightarrow A = 1,\ B = 2,\ C = -1$$

$$\Rightarrow \int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx = \int \left[x+1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}\right] dx$$

$$= \boxed{\frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C}$$

Ex) (case III) In general, for irreducible factors (quadratic) we will have $\frac{Ax+B}{ax^2+bx+c}$

Ex) $$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}$$

Example: $$\int \frac{2x^2-x+4}{x^3+4x}\,dx \Rightarrow \frac{2x^2-x+4}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$\Rightarrow A = 1,\ B = 1,\ C = -1$$

$$\Rightarrow = \int \left(\frac{1}{x} + \frac{x-1}{x^2+4}\right) dx$$

$$= \boxed{\ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C}$$

G) $\boxed{\text{Case IV}}$ $\displaystyle\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx$

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$\Rightarrow -x^3+2x^2-x+1 = A(x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+E)x$$
$$= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

$$\Rightarrow A=1,\ B=-1,\ C=-1,\ D=1,\ E=0$$

$$\Rightarrow = \int \left(\frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2}\right)dx$$

$$= \boxed{\ln|x| - \frac{1}{2}\ln(x^2+1) - \tan^{-1}(x) - \frac{1}{2(x^2+1)} + C}$$

Non Rational functions w/ substitution $\Rightarrow$ Partial Fractions

$\displaystyle\int \frac{\sqrt{x+4}}{x}\,dx$ Let $u=\sqrt{x+4} \Rightarrow u^2 = x+4$

$x = u^2-4$

$dx = 2u$

$$\Rightarrow \int \frac{\sqrt{x+4}}{x}\,dx = \int \frac{u}{u^2-4}\,2u\,du = 2\int \frac{u^2}{u^2-4}\,du$$

$$= 2\int \frac{u^2}{(u-2)(u+2)}\,du \qquad \frac{A}{u-2} + \frac{B}{u+2} = \frac{4}{(u-2)(u+2)}$$

$$= 2\int \left(1 + \frac{4}{(u+2)(u-2)}\right)du \qquad A(u+2) + B(u-2) = 4$$

$$A=1 \quad B=-1$$

$$= 2\int \left(1 + \frac{1}{u-2} + \frac{-1}{u+2}\right)du$$

$$= 2\left(u + \ln|u-2| - \ln|u+2|\right) + C = 2\left(\sqrt{x+4} + \ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right|\right) + C$$