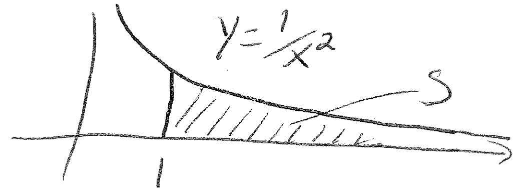
Section 8.8 - Improper Integrals

Consider the infinite region, S,



The area of $S = 1 - \frac{1}{t}$

Observe that $A(t) = 1 - \frac{1}{t}$ and $\lim_{t\to\infty} A(t) = \lim_{t\to\infty} 1 - \frac{1}{t} = 1$

The area, has finite value over an <u>infinite</u> interval.

In other words,

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$$
$$= \lim_{t\to\infty} 1 - \frac{1}{t} = 1$$

Improper Integrals of Type 1

If: (a) $\int_a^t f(x)\,dx$ exists for every $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided the limit exists

(b) $\int_t^b f(x)\,dx$ exists for every $t \leq b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided limit exists.

(c) If both (a), (b) are true with $a = b$, then

$$\int_{\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

If the limit <u>exists</u> $\Rightarrow$ integral is <u>convergent</u>

limit <u>does not exist</u> $\Rightarrow$ integral is <u>divergent</u>

Examples: $\int_1^\infty \frac{1}{x}\,dx$ $\quad \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

$\int_{-\infty}^{0} xe^x\,dx$ $\quad \int_1^\infty \frac{1}{x^p}\,dx$ (for which p?)

conv: $p>1$

div: $p \leq 1$

$\int_{-\infty}^{-1} e^{-2t}\,dt$

Improper Integrals Type 2

(a) f continuous on $[a,b)$, discontinuous at b

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx \quad \text{if lim exists}$$

(b) f cont. on $(a,b]$ disc. at a

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx \quad \text{if lim exists}$$

(c) f cont on (a,b) disc at c, $a<c<b$

$$\Rightarrow \int_a^b f(x)\,dx = \int_a^c f(x)\,dx \int_c^b f(x)\,dx$$

Ex) $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$ $\quad \int_0^{\pi/2} \sec(x)\,dx$ $\quad \int_0^3 \frac{dx}{x-1}$ (Try without imp.)

(Ans: $2\sqrt{3}$) $\quad$ (Ans: ∞ $\sec\to\infty$, $\tan\to\infty$) $\quad$ (Ans: $-\infty$)

$\int_0^1 \ln(x)\,dx$

Comparison Theorem

Suppose f and g are continuous s.t. $f(x) \geq g(x) \geq 0$ for $x \geq a$

(a) If $\int_a^\infty f(x)\,dx$ converges $\Rightarrow \int_a^\infty g(x)\,dx$ is convergent

(b) If $\int_a^\infty g(x)\,dx$ diverges $\Rightarrow \int_a^\infty f(x)\,dx$ is divergent

Ex) Show $\int_1^\infty e^{-x^2}\,dx$ is convergent

For $x \geq 1 \Rightarrow x^2 \geq x \Rightarrow -x^2 \leq -x \Rightarrow e^{-x^2} \leq e^{-x}$

do $\int_1^\infty e^{-x}\,dx$

Ex) Show $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is divergent $\left(\frac{1+e^{-x}}{x} > \frac{1}{x} \right)$

Ex) $\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\,dx$ $\left(\frac{\sec^2 x}{x\sqrt{x}} > \frac{1}{x^{3/2}} \right)$ divergent

$\sec(x) > 1$ for $0 < x < 1$

$\Rightarrow \sec^2(x) > 1$