Section 10.1 - Sequences

A sequence is a list of numbers written in a certain order. For example, $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$

or $\{a_1, a_2, \ldots, a_n, \ldots\}$ or $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

Sequences can be finite or infinite. The above examples are infinite sequences as there is no "last element", ie. the series does not terminate. A finite sequence has an integer number of terms, like $\{0,1,2,3,4,5\}$ or $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}\}$.

Examples of infinite sequence $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$, $a_n = \frac{n}{n+1}$

Write out term by term.

Given an infinite sequence of numbers, can we find the general term?

Ex) Find a formula for $\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \ldots\right\}$

$$a_n = (-1)^{n+1}\frac{n+2}{5^n}$$

Ex) Find a formula for $\{1, 1, 2, 3, 5, 8, 13, \ldots\}$ (Fibonacci sequence $\{f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2}$, $n \geq 3\}$)

Ex) Find a formula for $\{1, 1, 2, 6, 24, 120, 720, \ldots\}$ $\{n!\}_{n=1}^{\infty}$

Properties of factorials: Reduce $\frac{n!}{(n+1)!}$, $\frac{(n+1)!}{n!}$, $\frac{(n+3)!}{(n+1)!}$

$\frac{5!}{6!}$, $\frac{7!}{10!}$, $\frac{18!}{12!}$ etc; $\frac{(n-2)!}{(n+2)!}$

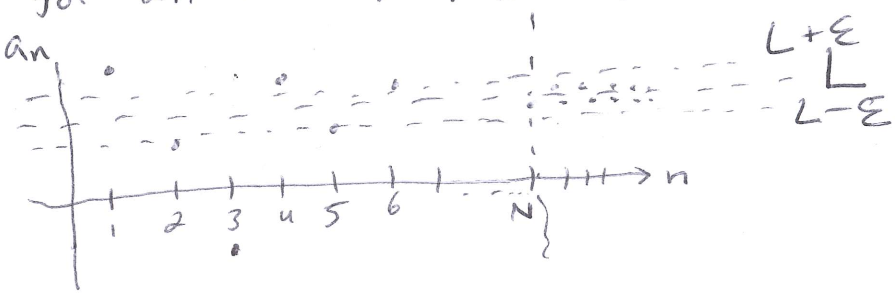
Definition: A sequence $\{a_n\}_{n=1}^{\infty}$ has a limit, say L, written

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n\to\infty,$$

if we can ~~take~~ make the a_n "as close" to L as we would like for n sufficiently large.

Definition: (Rigorous) A sequence $\{a_n\}_{n=1}^{\infty}$ has a limit L, if for every $\varepsilon > 0$ (constant), there exists an integer N s.t. for all $n \geq N$, $|a_n - L| < \varepsilon$

Picture:



Definition: If $\lim_{n\to\infty} a_n = \infty$, then there exists an integer N such that for $n \geq N$ and for every $M > 0$ we have that $a_n \geq M$.

Definition: If the $\lim_{n\to\infty} a_n$ exists and is finite, the sequence converges.
If the $\lim_{n\to\infty} a_n$ does not exists or is $\pm\infty$, the sequence diverges.

Properties: If $\{a_n\}$ and $\{b_n\}$ converge, then

$\{a_n\} \pm \{b_n\}$ converges

$\{a_n\} \cdot \{b_n\}$ converges

$\dfrac{\{a_n\}}{\{b_n\}}$ converges as long as $b_n \neq 0$

$\{ca_n\} = c\{a_n\}$ converges for a constant c

Squeeze Theorem: If $a_n \le b_n \le c_n$ for all $n \ge N$, N constant and the sequence $\{a_n\}$ and $\{c_n\}$ converge to L, ie. $\lim\limits_{n\to\infty} a_n = L$, $\lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$ so $\{b_n\}$ converges to L.

Corollary: If $\lim\limits_{n\to\infty} |a_n| = 0$ then $\lim\limits_{n\to\infty} a_n = 0$.

Examples: Do the following sequences converge?

(a) $a_n = \frac{n}{n+1}$, (b) $a_n = \frac{\ln(n)}{n}$, (c) $a_n = (-1)^n$, (d) $a_n = \frac{(-1)^n}{n}$

Challenge: $\{a_n\}_{n=1}^{\infty} = \left\{\frac{n!}{n^n}\right\}_{n=1}^{\infty}$ Idea: Use squeeze theorem

$n!$ is certainly always positive and $n^n > 0$ for $n \ge 1$

$\Rightarrow \frac{n!}{n^n} > 0$. Now, we can write out the n^{th} term

$$\frac{n!}{n^n} = \frac{1}{n} \cdot \underbrace{\left(\frac{2 \cdot 3 \cdot 4 \cdot 5 \cdots n}{n \cdot n \cdot n \cdot n \cdots n}\right)}_{<1} < \frac{1}{n}$$

Why? $\frac{2}{n} < 1$, $\frac{3}{n} < 1, \ldots, \frac{n}{n} = 1$

$\Rightarrow$ the products must be less than 1

$\Rightarrow \frac{n!}{n^n} < \frac{1}{n}$

So we have $0 \le \frac{n!}{n^n} \le \frac{1}{n}$. Let $b_n = 0$, $c_n = \frac{1}{n}$

s.t. $b_n \le a_n \le c_n$, and $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} 0 = 0$

$\lim\limits_{n\to\infty} c_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$

So by squeeze theorem, $\lim\limits_{n\to\infty} \frac{n!}{n^n} = 0$ so $\left\{\frac{n!}{n^n}\right\}_{n=1}^{\infty} \to 0$. Convergent.

Definition: A sequence of the form $\{r^n\}$ where $r =$ constant is called a <u>geometric sequence</u>.

The sequence $\{r^n\}$ <u>converges</u> for $-1 < r \le 1$ or $|r| < 1$ and $r = 1$

<u>Definition</u>: We say a sequence is <u>increasing</u> if $a_n < a_{n+1}$ $\forall n \geq 1$

'' '' <u>decreasing</u> if $a_n > a_{n+1}$ $\forall n \geq 1$

A sequence is <u>monotonic</u> if it is increasing <u>or</u> decreasing

Ex) Show $a_n = \dfrac{n}{n^2+1}$ is decreasing for $n \geq 1$

<u>Solution</u>: We want to show $\dfrac{n+1}{(n+1)^2+1} < \dfrac{n}{n^2+1}$

The above is true if and only if (iff)

$(n+1)(n^2+1) < n[(n+1)^2+1]$ since $n \geq 1$

$\iff n^3+n^2+n+1 < n^3+2n^2+2n$

$\iff 1 < n^2+n$ which is always true since $n \geq 1$

<u>Definition</u>: A sequence is <u>bounded above</u> if $\exists$ a number M such that $a_n \leq M$ for all $n \geq 1$

A sequence is <u>bounded below</u> if $\exists$ a number m such that $m \leq a_n$ for all $n \geq 1$

A sequence is <u>bounded</u> if both hold, $m \leq a_n \leq M$ for all $n \geq 1$

<u>Theorem</u>: Every bounded monotone sequence is convergent.

<u>Example</u>: Let $\{a_n\}$ be defined as $a_1 = 2$, $a_{n+1} = \frac{1}{2}(a_n+6)$ for $n = 1, 2, 3, \ldots$

To use the Theorem, we must show $\{a_n\}$ is monotone and bounded.

<u>Monotone</u>: By induction: Base Case $a_2 = 4 > 2 = a_1$ ✓

<u>Assumption</u>: $a_{k+1} > a_k$ $n=k$ assumption step

Now prove for $n = k+1$

From assumption step, $a_{k+1} > a_k$

$$a_{k+1} + 6 > a_k + 6$$

$$\frac{1}{2}(a_{k+1} + 6) > \frac{1}{2}(a_k + 6)$$

$$a_{k+2} > a_{k+1}$$ by initial formula definition of an

So by induction, $\{a_n\}$ is monotonic as $\{a_n\}$ is only increasing

Bounded: We will show $a_n < 6$ for all n. Note $a_n > 0 = m$ since we cannot get negative entries by adding 6 and halving the result. Or see that $a_1 = 2 < a_2 < a_3 < \ldots$ since a_n is increasing so $m = 2$ is another choice.

Again we use induction.

Base Case: $a_1 = 2 < 6$

Assumption: $a_k < 6$

Prove for $n = k+1$: $a_k < 6 \Rightarrow a_k + 6 < 12$

$$\Rightarrow \frac{1}{2}(a_k + 6) < \frac{1}{2} \cdot 12$$

$$\Rightarrow a_{k+1} < 6$$

So by induction, $a_n < 6$ for all n.

So by Theorem, $\{a_n\}$ is convergent. ∎

Note: The Theorem does not tell us the limit, just that it exists.

Call the limit L. Then

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n + 6) = \frac{1}{2}\left(\lim_{n\to\infty} a_n + 6\right) = \frac{1}{2}(L+6)$$

Since as $n \to \infty$, $a_n \to L$ and $a_{n+1} \to L$ since $\{a_n\}$ convergent

$$\Rightarrow L = \frac{1}{2}(L+6) \Rightarrow \frac{1}{2}L = 3 \Rightarrow L = 6$$