Section 10.2 - Series

When we add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$

we get the expression

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n$$

Does this expression make sense? This section and the next 4 attempt to answer that question.

Examples: ① $1+2+3+4+\cdots+n+\cdots$

If we add up the first n terms we would get

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \to \infty \quad \text{as } n \to \infty$$

Look at the cumulative sum (a sequence!)

$$\{1, 3, 6, 10, 15, 21, \ldots\} \to \infty \text{ as } n \to \infty$$

② $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots$

Look at the cumulative sum (partial sums)

$$\left\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \frac{127}{128}, \ldots\right\} \Rightarrow 1 - \frac{1}{2^n} \to 1 \text{ as } n \to \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n} = 1, \text{ a finite number}$$

Definition: Given the series $\sum_{n=1}^{\infty} a_n$, $S_n = \sum_{i=1}^{n} a_i$ is the n^{th} partial sum

We can view the partial sums as their own sequence

$S_1 = a_1$, $S_2 = a_1 + a_2$, $S_3 = a_1 + a_2 + a_3$, ..., $S_n = \sum_{i=1}^{n} a_i$

$$\Rightarrow S_n = \{S_1, S_2, S_3, S_4, \ldots, S_n\}$$

Definition: Given a series $\sum_{n=1}^{\infty} a_n$, if $\{S_n\}$, the sequence of partial sums, is convergent, and $\lim_{n\to\infty} S_n = L$ exists and is a real number, then $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} a_n = L$

Otherwise, $\sum_{n=1}^{\infty} a_n$ is divergent.

Note: $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i = \lim_{n\to\infty} S_n$

Geometric Series

$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots$ is convergent

if $|r|<1$, and we get $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ for $|r|<1$

* If $|r| \geq 1$, $\Rightarrow$ series is divergent

Example: $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \ldots$ Is the series convergent or divergent?

$a = 5$, $r = \frac{-\frac{10}{3}}{5} = -\frac{10}{3} \cdot \frac{1}{5} = -\frac{2}{3}$ $\Rightarrow r = -\frac{2}{3} \Rightarrow |r| = \frac{2}{3} < 1$

$r = \frac{\frac{20}{9}}{-\frac{10}{3}} = \frac{20}{9} \cdot \frac{-3}{10} = -\frac{2}{3}$ $\Rightarrow$ Convergent

$\Rightarrow \sum_{n=1}^{\infty} 5\left(-\frac{2}{3}\right)^{n-1} = \frac{5}{1-\left(-\frac{2}{3}\right)} = \frac{5}{\frac{5}{3}} = 3$

Example: Is $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ convergent or divergent?

$= \sum_{n=1}^{\infty} (2^2)^n 3^{-(n-1)} = \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1}$

$\Rightarrow$ divergent since $|r| = \frac{4}{3} \geq 1$

Example: Find sum of $\sum_{n=0}^{\infty} x^n$ for $|x|<1$

$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots = \sum_{n=1}^{\infty} x^{n-1}$ $\Rightarrow$ $a = 1$, $r = x \Rightarrow |r| = |x| < 1$

$= \frac{1}{1-x}$

Note: To reindex if you add in the bottom of Σ, you must subtract the same amount from each n in a_n.

If you subtract in the bottom of Σ, you must add the same amount in each n in a_n.

Example: Show $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges. Find its sum.

$$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) \quad \text{by partial fractions}$$

$$= \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right) \cancel{+} \cdots \cancel{+} \left(\cancel{\tfrac{1}{n-1}} - \cancel{\tfrac{1}{n}}\right) + \left(\cancel{\tfrac{1}{n}} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}, \quad \text{so } \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n} = 1 \Rightarrow \text{Convergent}$$

This is called a telescoping sum.

Example: Show $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ diverges (harmonic series)

Proof:

$$S_1 = 1$$
$$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$$
$$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2} = 2$$
$$S_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{3}{2} = \frac{5}{2}$$

$S_{16} > 1 + \frac{4}{2} = 3$ and so on

$\Rightarrow S_{2^n} > 1 + \frac{n}{2} \Rightarrow$ as $n \to \infty$ $S_{2^n} \to \infty$

$\Rightarrow \{S_n\}$ diverges $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n}$ diverges ■

Theorem: If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

The converse is not true, in general. $\lim_{n\to\infty} a_n = 0$ is a necessary but not sufficient condition.

Test for Divergence: If $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

NOTE: If $\lim_{n\to\infty} a_n = 0$ this test tells you nothing !!!

Rules: If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent, then

① $\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$

② $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

And the RHS of the above (and LHS) are convergent.

Example: Find $\sum_{n=1}^{\infty} \left(\frac{1}{n(n+1)} + \frac{1}{2^n} \right)$

Note: A finite number of terms does not affect convergence.

Example: Find a rational number $\frac{p}{q}$ for $p, q \in \mathbb{Z}$ for the decimal expansion. $3.\overline{417} = 3.417417417417\ldots$

Solution: $3.\overline{417} = 3 + 0.417 + 0.000417 + 0.000000417$

$$= 3 + \frac{417}{10^3} + \frac{417}{10^6} + \frac{417}{10^9} + \frac{417}{10^{12}} + \cdots$$

$$= 3 + \sum_{n=1}^{\infty} \frac{417}{10^{3n}} = 3 + 417 \sum_{n=1}^{\infty} \left(\frac{1}{10^3}\right)^n$$

$$= 3 + 417 \sum_{n=1}^{\infty} \frac{1}{10^3} \left(\frac{1}{10^3}\right)^{n-1} \qquad a = \frac{1}{10^3} \quad r = \frac{1}{10^3}$$

$$= 3 + 417 \frac{\frac{1}{10^3}}{1 - \frac{1}{10^3}} = 3 + \frac{417}{10^3} \cdot \frac{10^3}{999}$$

$$= 3 + \frac{417}{999} = \frac{2997 + 417}{999} = \frac{3414}{999} = \frac{1138}{333}$$