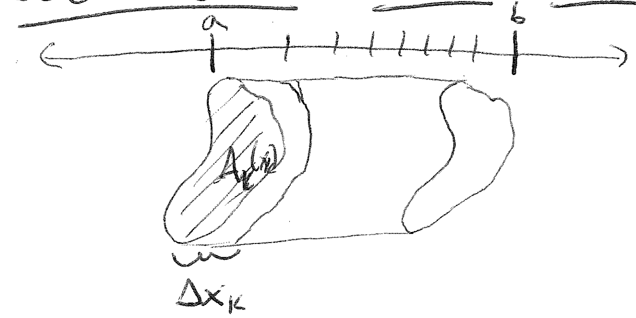
Section 6.1 - Volumes Using Cross Sections, Disks, Washers



From geometry, we know a cylindrical shape has volume

$$V = (\text{Area of base}) \cdot (\text{height}) = A \cdot h$$

If the cross section of the solid S at each point of x in the interval $[a, b]$, then the area of the cross section is a function of x, eg. $A(x)$,

So we partition $[a, b]$ into n pieces such that $a = x_0 < x_1 < \dots < x_n = b$ so that the volume of the k^{th} slice is

$$V_K = A(x_K)\Delta x_K$$

Then we sum up the $V_K \Rightarrow V \approx \sum_{K=1}^{n} V_K = \sum_{K=1}^{n} A(x_K)\Delta x_K$

These are the Riemann sums for the solid. To find the actual area we need to take a limit!!

$$\Rightarrow \lim_{n\to\infty} \sum_{K=1}^{n} A(x_K)\Delta x_K = \int_a^b A(x)\,dx$$

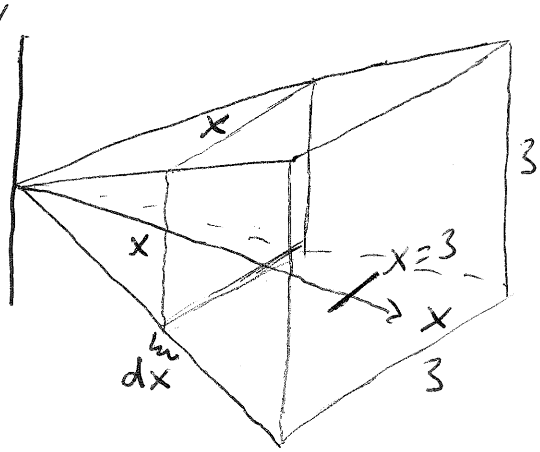
This is our definition:

The <u>Volume</u> ~~of~~ using cross sections of a solid that has cross-sectional area $A(x)$ is given by

$$\boxed{V = \int_a^b A(x)\,dx}$$

Examples:

① A 3m high pyramid with a square 3m x 3m base is arranged along the x-axis. Find its volume.



Each cross section of the pyramid is an x by x ~~Sequence~~ square that increases in size from $x=0$ to $x=3$.

$\Rightarrow A(x) = x \cdot x = x^2$

from $a=0$ to $b=3$

$$\Rightarrow V = \int_a^b A(x)\,dx = \int_0^3 x^2\,dx = \frac{x^3}{3}\Big|_0^3 = \frac{27}{3} - 0 = 9$$

$$\boxed{V = 9m^3}$$

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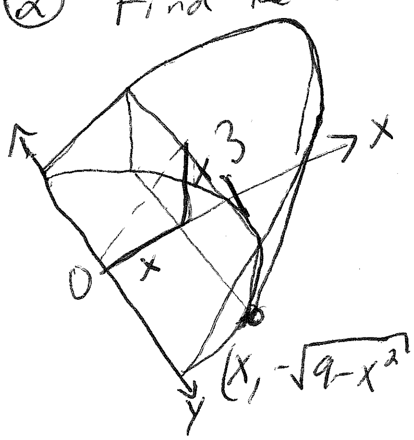
② Find the volume of the curved wedge, where the base is half a circle with radius 3 and a plane at 45°



Solution: From the picture, we can deduce the cross-sections are rectangles.

Each rectangle has height $= x$ (45° angle, same side lengths)

Circle is $x^2 + y^2 = 9 \Rightarrow y = \pm\sqrt{9-x^2}$

So $A(x) = (h)(w) = (x)(2\sqrt{9-x^2})$

$= 2x\sqrt{9-x^2}$

x ranges 0 to 3

$$\Rightarrow V = \int_a^b A(x)\,dx = \int_0^3 2x\sqrt{9-x^2}\,dx = -\frac{2}{3}(9-x^2)^{3/2}\Big|_0^3$$

$$= \frac{2}{3}(9)^{3/2} = \boxed{18}$$

Disk Method

Solids of revolution are generated by revolving a portion of a plane region about either the x or y axis. The region *must* border or cross the axis of redution to ~~have~~ be able to use the disk method.

Volume by Disks, Rotation about x-axis

$$V = \int_a^b A(x)\,dx = \int_a^b \pi\left[R(x)\right]^2 dx$$

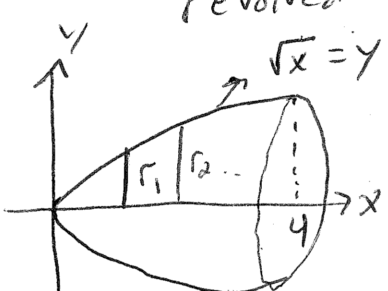
Volume by Disks, Rotation about y-axis

$$V = \int_c^d A(y)\,dy = \int_c^d \pi\left[R(y)\right]^2 dy$$

Similar to cross sections but they are all circles $\Rightarrow A = \pi R^2$, but all a function of x so $A(x) = \pi(R(x))^2$

Examples

① The region between $y=\sqrt{x}$ for $0 \le x \le 4$ and x-axis revolved around the x-axis. Find its volume.

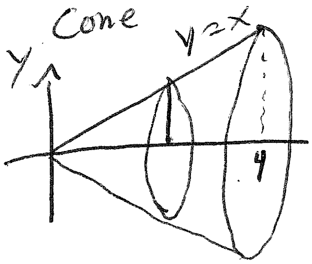


$$V = \int_a^b A(x)\,dx, \quad R(x) = \sqrt{x}$$

$$= \int_0^4 \pi\left(\sqrt{x}\right)^2 dx = \int_0^4 \pi x\,dx$$

$$= \pi\int_0^4 x\,dx = \pi \left.\frac{x^2}{2}\right|_0^4 = \boxed{8\pi}$$

(2) The region between the curve $y = x$ and $0 \le x \le 4$ and the x-axis. Revolve around x-axis. Find volume



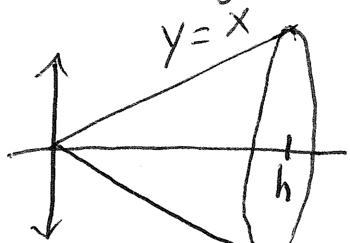
$$V = \int_a^b A(x)\,dx \qquad A(x) = \pi(R(x))^2 = \pi x^2$$

$$V = \pi \int_0^4 x^2\,dx$$

$$V = \pi \frac{x^3}{3}\Big|_0^4 = \frac{64}{3}\pi$$

Formula: $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi 4^2 \cdot 4 = \frac{64}{3}\pi$ ✓ same.

In general:



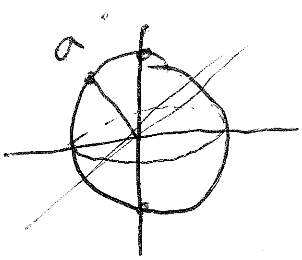
$$V = \int_0^h \pi x^2\,dx$$

$$V = \frac{1}{3}\pi h^3 = \frac{1}{3}\pi h^2 \cdot h$$

for when $r = h$

(3) Rotate the circle $x^2+y^2=a^2$ about the x-axis to generate a sphere. Find the volume.

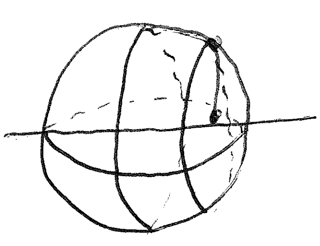
Solution: We know we should get $V = \frac{4}{3}\pi a^3$ by the usual formula.



Slices are all circles, so we have

$$x^2+y^2=a^2 \Rightarrow y^2 = a^2 - x^2 \qquad y = \pm\sqrt{a^2-x^2}$$

$$A(x) = \pi(y(x))^2 = \pi\left(\sqrt{a^2-x^2}\right)^2$$



$$\Rightarrow A(x) = \pi(a^2-x^2)$$

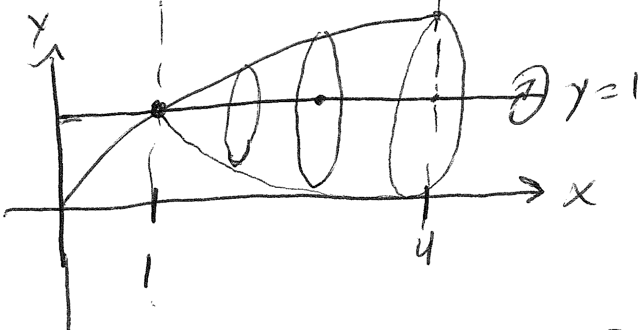
$$\Rightarrow V = \int_{-a}^{a} \pi(a^2-x^2)\,dx = \pi\left(a^2x - \frac{x^3}{3}\right)\Big|_{-a}^{a}$$

$$= \pi\left[a^3 - \frac{a^3}{3} - \left(-a^3 + \frac{a^3}{3}\right)\right]$$

$$= \pi\left[2a^3 - \frac{2}{3}a^3\right] = \boxed{\frac{4}{3}\pi a^3}$$

③

(3) Find volume of revolving the region bounded by $y=\sqrt{x}$, $y=1$, $x=4$ about the line $y=1$



So x ranges $1 \le x \le 4 \Rightarrow a=1, b=4$

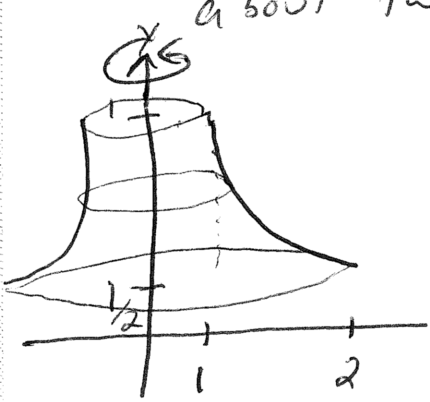
$$V = \int_a^b \pi (R(x))^2 \, dx$$

How to find $R(x)$? at $x=4$, we know $\sqrt{x} = \sqrt{4} = 2$, but radius $= 1$ by the picture. $\Rightarrow R(x) = \sqrt{x} - 1$

$$\Rightarrow \quad V = \int_1^4 \pi (R(x))^2 \, dx = \int_1^4 \pi (\sqrt{x}-1)^2 \, dx$$

$$= \pi \left[\frac{x^2}{2} - \frac{4}{3} x^{3/2} + x \right] \Big|_1^4 = \boxed{\frac{7\pi}{6}}$$

(4) Find volume of Solid obtained by revolving $y = \frac{1}{x}$ from $x=1$ to $x=2$ about the y-axis



Formula is $V = \int_c^d \pi (R(y))^2 \, dy$

So y ranges $\frac{1}{2} \le y \le 1$

but we have $R(y)$ now! $\Rightarrow \underline{R(y) = \frac{1}{y}}$

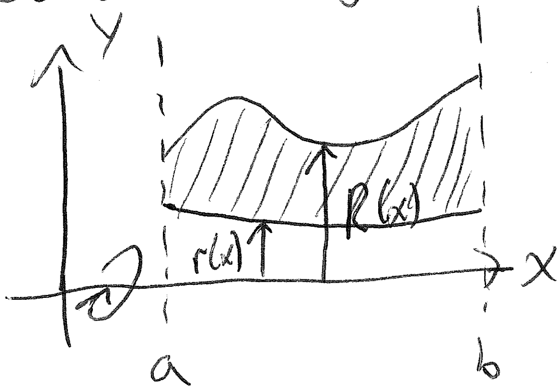
$$\Rightarrow \quad V = \int_{1/2}^1 \pi \frac{1}{y^2} \, dy = -\pi \frac{1}{y} \Big|_{1/2}^1$$

$$= -\frac{\pi}{1} - \left(-\pi \frac{1}{\frac{1}{2}}\right) = \boxed{\pi}$$

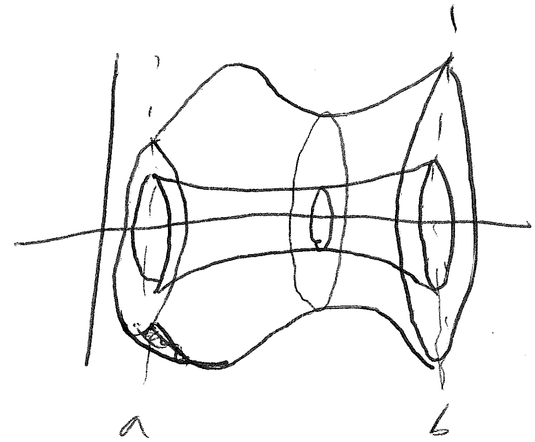
Washer Method

For the washer method, we have a "hole" in the area cross-section, so there is an "outer" and "inner" radius.

So we'll be given the area between two functions, say



rotate about x-axis $\Longrightarrow$



Washers Formulas

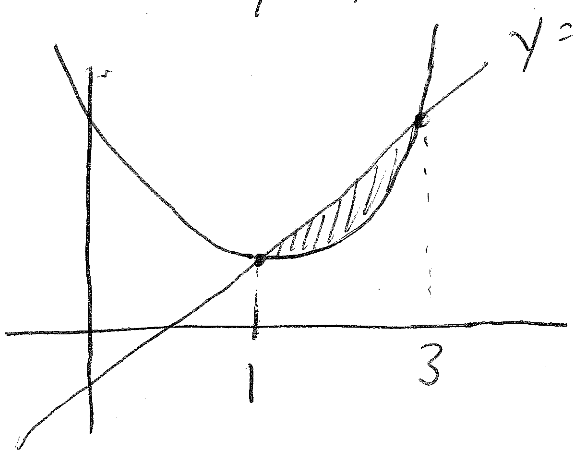
$$V = \int_a^b A(x)\,dx = \int_a^b \pi\left[(R(x))^2 - (r(x))^2\right]dx$$

$$V = \int_c^d A(y)\,dy = \int_c^d \pi\left[(R(y))^2 - (r(y))^2\right]dy$$

"Outer radius minus inner radius"

Examples

① Find volume formed by rotating region bounded by $y = x^2 - 2x + 2$ and $y = 2x - 1$



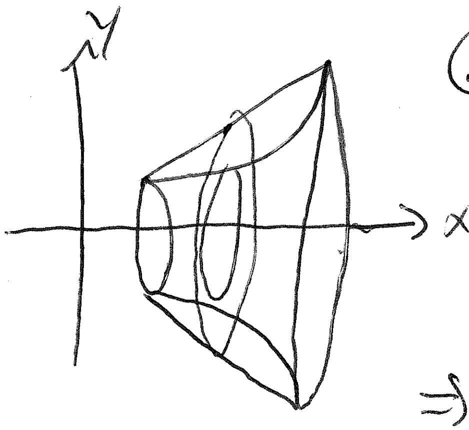
① Set equations equal to find x coordinates of intersection

$x^2 - 2x + 2 = 2x - 1$

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$x = 3, 1$



② Outer radius (line) $R(x) = 2x-1$

Inner radius (parabola) $r(x) = x^2-2x+2$

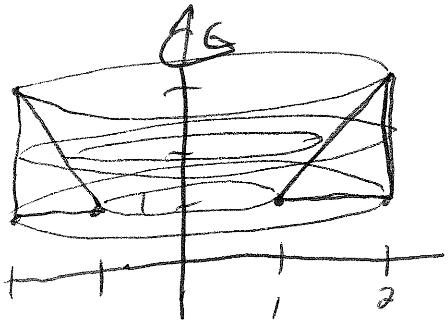
$$\Rightarrow V = \int_1^3 \pi\left((2x-1)^2 - (x^2-2x+2)^2\right) dx$$

$$= \pi\int_1^3 \left(-x^4+4x^3-4x^2+4x-3\right) dx$$

$$= \pi\left[-\frac{1}{5}x^5 + x^4 - \frac{4}{3}x^3 + 2x^2 - 3x\right] dx$$

$$= \boxed{\frac{104}{15}\pi}$$

② Find the ~~area~~ volume of the ~~region~~ solid by rotating the triangular region with vertices at (1,1), (2,1), (2,3) about the y-axis.



Linear part is $y = 2x-1$

$$\Longrightarrow \frac{1}{2}(y+1) = x$$

Where is y ranging from? $1 \le y \le 3$

Outer radius is always 2

Inner is $\frac{1}{2}(y+1)$

$$\Rightarrow V = \int_1^3 \pi\left(2^2 - \left(\tfrac{1}{2}(y+1)\right)^2\right) dy$$

$$\Rightarrow V = \pi\int_1^3 \left(-\frac{1}{4}y^2 - \frac{1}{2}y + \frac{15}{4}y\right) dy$$

$$\Rightarrow \boxed{V = \frac{10}{3}\pi}$$