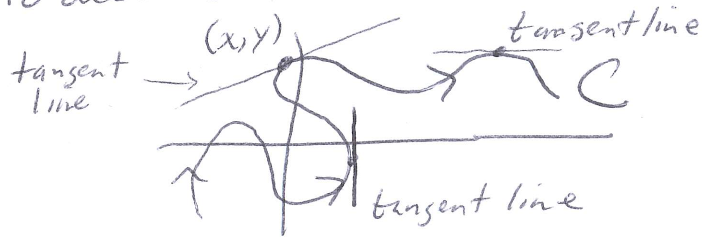
Section 11.2 - Calculus on Parametric Curves

Given $x = f(t)$ and $y = g(t)$, how can we find $\frac{dy}{dx}$ in the xy-plane?

Recall $x = f(t)$ and $y = g(t)$ is a way to describe some arbitrary curve in the xy-plane



$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \text{ if } \frac{dx}{dt} \neq 0. \ (\ast) \text{ How?}$$

Let $y = F(x) \Rightarrow g(t) = F(f(t))$ since $y = g(t),\ x = f(t)$ parametric eqns.

$\Rightarrow g'(t) = F'(f(t)) \cdot f'(t)$ Chain rule

$(\dagger) \Rightarrow F'(f(t)) = \dfrac{g'(t)}{f'(t)}$

$\Rightarrow F'(x) = \dfrac{dy/dt}{dx/dt} \Rightarrow \boxed{\dfrac{dy}{dx} = \dfrac{dy/dt}{\frac{dx}{dt}}}$ since $y = F(x)$

From $(\ast)$, we can see that we have

Horizontal tangents when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$

Vertical tangents when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$

Indeterminate when $\frac{dy}{dt} = \frac{dx}{dt} = 0 \Rightarrow \frac{dy}{dx} = \frac{0}{0}$!!

What about the second derivative? Start with $(\dagger)$ above

$$F'(f(t)) = \frac{g'(t)}{f'(t)} \Rightarrow \frac{d}{dt}\left(F'(f(t))\right) = \frac{d}{dt}\left(\frac{g'(t)}{f'(t)}\right)$$

$$\Rightarrow F''(f(t))f'(t) = \frac{d}{dt}\left(\frac{g'(t)}{f'(t)}\right)$$

Recall, that

$g'(t) = \frac{dy}{dt}$, $f'(t) = \frac{dx}{dt} \Rightarrow \frac{g'(t)}{f'(t)} = \frac{dy}{dx}$

$$\Rightarrow F''(f(t)) = \frac{1}{f'(t)} \frac{d}{dt}\left(\frac{g'(t)}{f'(t)}\right)$$

$$\Rightarrow F''(x) = \frac{1}{\frac{dx}{dt}} \frac{d}{dt}\left(\frac{dy}{dx}\right)$$

$$\Rightarrow \boxed{\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}}$$

Ex) Define a curve C as $\begin{array}{l} x(t) = t^2 \\ y(t) = t^3 - 3t \end{array}$

a) Show C has two tangents at $(3,0)$

b) Find where there are horizontal and vertical asymptotes

c) Determine concavity

d) Sketch.

Solution: (a) At which t do we reach coordinate $(3,0)$?

$y = 0 \Rightarrow \quad 0 = t^3 - 3t \Rightarrow t(t^2-3) = 0 \Rightarrow \cancel{t=0},\ t = \pm\sqrt{3}$

but $x(t) = t^2$ so only $t = \pm\sqrt{3}$

$$\frac{dy}{dx} = \frac{3t^2-3}{2t} = \frac{3}{2}\left(t - \frac{1}{t}\right) = \pm\frac{6}{2\sqrt{3}} = \pm\sqrt{3}$$

So tangent lines are $\begin{array}{l} y = \sqrt{3}(x-3) \\ y = -\sqrt{3}(x-3) \end{array}$

(b) Horiz. tangents when $\frac{dy}{dt} = 0$, $\frac{dx}{dt} \neq 0$ so $3t^2 - 3 = 0 \Rightarrow t = \pm 1$, check $\frac{dx}{dt} = 2t = \pm 2 \neq 0$ ✓

$\Rightarrow$ coordinates are $\begin{array}{l} t = -1 \Rightarrow (1,2) = (x,y) \\ t = 1 \Rightarrow (1,-2) \end{array}$

Vertical tangents when $\frac{dx}{dt} = 0$, $\frac{dy}{dt} \neq 0$ so $t^2 = 0 \Rightarrow t = 0$ check $\frac{dy}{dt} = 3t^2 - 3 = -3 \neq 0$ ✓

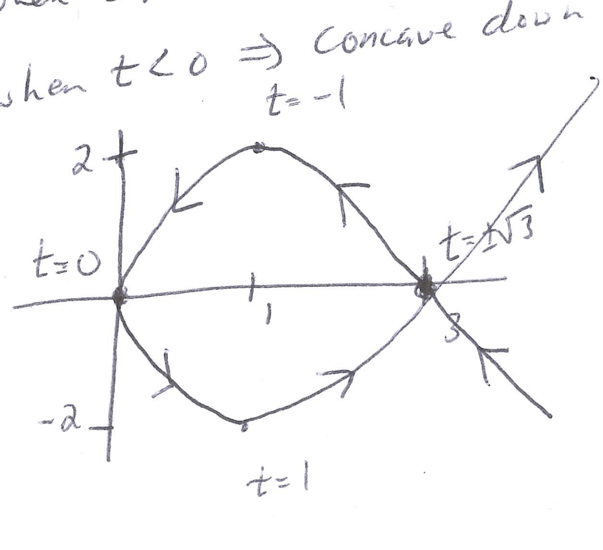
$t = 0 \Rightarrow \boxed{(0,0)}$

(c) $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{3}{2}\left(1 + \frac{1}{t^2}\right)}{2t} = \frac{3\left(1 + \frac{1}{t^2}\right)}{4t} = 3 \cdot \frac{1}{t^2} \cdot \frac{(t^2+1)}{4t} = \frac{3(t^2+1)}{4t}$$

So, we have $\frac{d^2y}{dx^2} > 0$ when $t > 0 \Rightarrow$ concave up for $t > 0$

$\frac{d^2y}{dx^2} < 0$ when $t < 0 \Rightarrow$ concave down for $t < 0$

(d) Combine (a)-(c)



- Can you eliminate the parameter?

$x = t^2 \Rightarrow t = \pm\sqrt{x}$

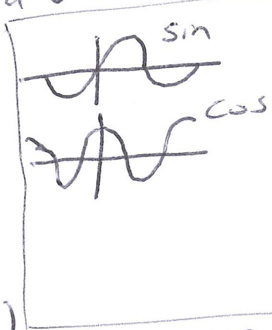
$y = t^3 - 3t \Rightarrow y = t(t^2-3)$

$y = \pm\sqrt{x}(x-3)$

$\Rightarrow y^2 = x(x-3)^2$ looks familiar??

Ex) Find tangent line to $x = a(t - \sin(t))$, $y = a(1-\cos(t))$ at $t = \pi/3$

Also find points where tangent line is horiz. and vertical.



Solution: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\sin(t)}{a(1-\cos(t))} = \frac{\sin(t)}{1-\cos(t)}$

$\Rightarrow \left.\frac{dy}{dx}\right|_{t=\pi/3} = \frac{\sin(\pi/3)}{1-\cos(\pi/3)} = \frac{\sqrt{3}/2}{1-1/2} = \sqrt{3}$

$x = a\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$

$y = a(1-\cos(\pi/3)) = \frac{a}{2}$

$y - y_0 = m(x-x_0)$

$\Rightarrow \boxed{y - \frac{a}{2} = \sqrt{3}\left(x - a\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right)}$

Horizontal tangents: where $\sin t = 0$, $\cos t \neq 1$

$\Rightarrow$ odd integers times π or $(2n-1)\pi = t$ $n \in \mathbb{Z}$

$\Rightarrow$ & when $t = (2n-1)\pi$

So $(x,y) = ((2n-1)\pi a, 2a)$ have Horiz. Tangents

Vertical tangents: where $1-\cos(t) = 0 \Rightarrow \cos(t) = 1$

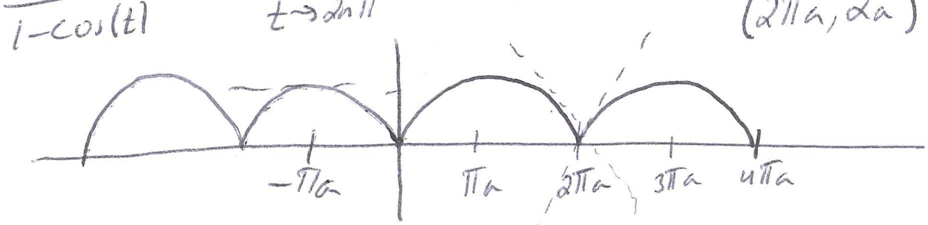
$\Rightarrow t = 2n\pi$ for $n \in \mathbb{Z}$

but $\frac{dy}{dx} = \frac{\sin(t)}{1-\cos(t)} = \frac{\sin(2n\pi)}{1-\cos(2n\pi)} = \frac{0}{0}$!!! need limits

$\lim_{t\to 2n\pi^-} \frac{dy}{dx} = \lim_{t\to 2n\pi^-} \frac{\sin(t)}{1-\cos(t)} \stackrel{L'H}{=} \lim_{t\to 2n\pi^-} \frac{\cos(t)}{\sin(t)} = -\infty$

$\lim_{t\to 2n\pi^+} \frac{dy}{dx} = \lim_{t\to 2n\pi^+} \frac{\sin(t)}{1-\cos(t)} \stackrel{L'H}{=} \lim_{t\to 2n\pi^+} \frac{\cos(t)}{\sin(t)} = +\infty$

not same, but okay vertical at $(2\pi a, 2a)$



Areas

Recall for functions, the area under the curve is given by $A = \int_a^b F(x)\,dx = \int_a^b y\,dx$ for $y = F(x)$.

For parametric equations, $x = f(t)$, $y = g(t)$

$$\Rightarrow \frac{dx}{dt} = f'(t) \Rightarrow dx = f'(t)\,dt$$

$$\Rightarrow A = \int_a^b y\,dx = \int_\alpha^\beta g(t)\,f'(t)\,dt$$

Ex) Find the area under one arch of $x = a(t - \sin(t))$, $y = a(1 - \cos(t))$

Solution

We know from previous problem the first arch is from $x = 0$ to $x = 2\pi a$, so

$$A = \int_{x=0}^{x=2\pi a} y\,dx = \int_0^{2\pi} a(1-\cos(t))\cdot a(1-\cos(t))\,dt$$

$$= a^2 \int_0^{2\pi} 1 - 2\cos(t) + \cos^2(t)\,dt$$

$$= a^2 \int_0^{2\pi} \left(1 - 2\cos(t) + \frac{1}{2} + \frac{1}{2}\cos(2t)\right) dt$$

$$= a^2\left(\frac{3}{2}t - 2\sin(t) + \frac{1}{4}\sin(2t)\right)\Big|_0^{2\pi}$$

$$= \boxed{3\pi a^2}$$

if $x = 0$
$0 = a(t - \sin(t))$
$\Rightarrow t = 0$
if $x = 2\pi a$
$2\pi a = a(t - \sin(t))$
$2\pi = t - \sin(t)$
$t = 2\pi$

Arc Lengths / Areas of Surfaces

Recall the Arc Length formula

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$ Now using parametric equations,

$$L = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt} \cdot dt = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Ex) Find the arc lengths for

(a) $\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases}$ $0 \le t \le 2\pi$

(b) $\begin{cases} x = \sin(2t) \\ y = \cos(2t) \end{cases}$ $0 \le t \le 2\pi$

(c) $\begin{cases} x = a(t - \sin(t)) \\ y = a(1 - \cos(t)) \end{cases}$ $0 \le t \le 2\pi$

For (c), Note: $a\int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta = a\int_0^{2\pi} \sqrt{4\sin^2\left(\frac{\theta}{2}\right)}\, d\theta$ by trick

$\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$

$\Rightarrow \sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1 - \cos(\theta))$

$4\sin^2\left(\frac{\theta}{2}\right) = 2(1 - \cos(\theta))$

$\sin\left(\frac{\theta}{2}\right) > 0$, $0 \le \theta \le 2\pi$

$$= 2a\int_0^{2\pi} \left|\sin\left(\frac{\theta}{2}\right)\right| d\theta$$

$$= 2a\int_0^{2\pi} \sin\left(\frac{\theta}{2}\right) d\theta = \underline{\underline{8a}}$$

Surface Area is similar $S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Ex) Rotate $\begin{cases} x = r\cos(t) \\ y = r\sin(t) \end{cases}$ $0 \le t \le \pi$ about x-axis