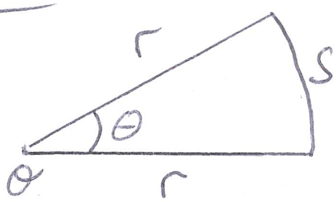
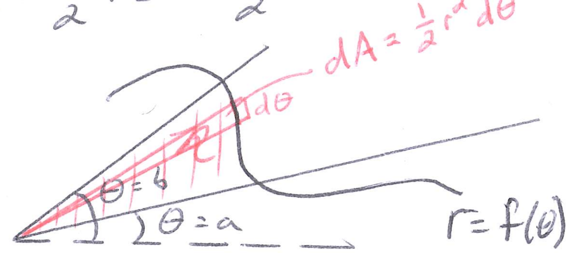
Section 11.5 - Areas and Lengths in Polar Coordinates



Recall that the area of a sector is given by

arc length, $s = r\theta$

$$A = (\text{total area})(\text{ratio of } s \text{ to } C = 2\pi r) = \pi r^2 \cdot \frac{s}{2\pi r} = \frac{1}{2} r s = \frac{1}{2} r^2 \theta.$$



For a general area, call it $\mathcal{R}$, we have the picture to the right.

We can approximate $\mathcal{R}$ by slices that look like the sector in the top figure for some thickness $d\theta$.

So the area of $\mathcal{R}$ is approximately

$$A \approx \sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2}(f(\theta_k))^2 \Delta\theta_k$$

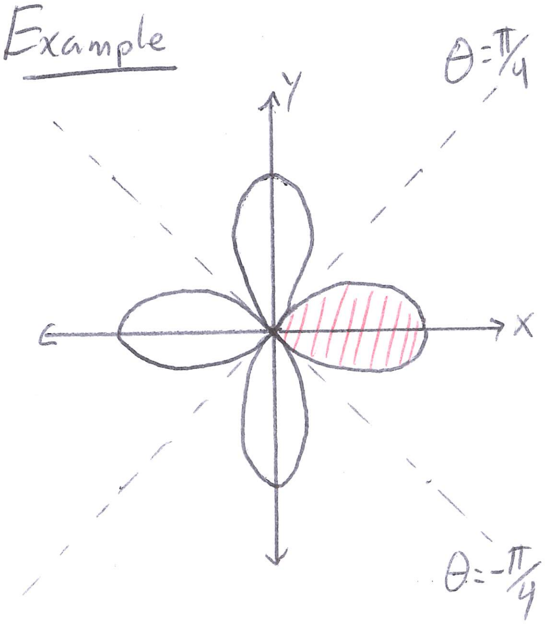
the partition is finer, so we get more sectors

assuming f is continuous $\Rightarrow A = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2}(f(\theta_k))^2 \Delta\theta_k$

$$\Rightarrow A = \int_a^b \frac{1}{2}(f(\theta))^2 \, d\theta$$

We can express this as well using differentials $dA = \frac{1}{2} r^2 d\theta \Rightarrow \int_a^b dA = \int_a^b \frac{1}{2} r^2 d\theta.$

Example



Find the area enclosed by one loop of $r = \cos(2\theta)$

Solution: Picture from last lecture.

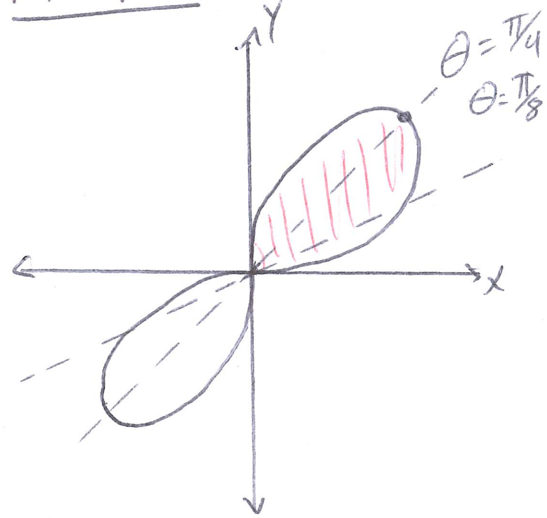
For $\theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]$, we get one petal.

$$A = \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2(2\theta) \, d\theta$$

$$\overset{\text{Symmetry}}{=} \int_0^{\pi/4} \cos^2(2\theta) \, d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{2} + \frac{1}{2}\cos(4\theta) \, d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{8}\sin(4\theta) \Big|_0^{\frac{\pi}{4}} = \boxed{\frac{\pi}{8}}$$

Example: Find the area of one petal of $r^2 = \sin(2\theta)$



Solution: Picture from lecture.

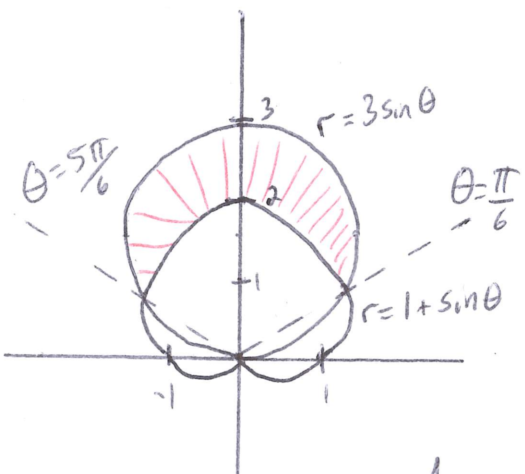
For $\theta \in [0, \frac{\pi}{2}]$, we get petal in quadrant I

$$A = \int_0^{\pi/2} \frac{1}{2} r^2 \, d\theta = \frac{1}{2}\int_0^{\pi/2} \sin(2\theta)\, d\theta$$

$$= -\frac{1}{4}\cos(2\theta)\Big|_0^{\pi/2} = -\frac{1}{4}\cos(\pi) - \left(-\frac{1}{4}\cos(0)\right)$$

$$= \frac{1}{4} + \frac{1}{4} = \boxed{\frac{1}{2}}$$

Example: Find the area ~~outside~~ inside the circle $r = 3\sin\theta$ and ~~in~~ outside the cardoid $r = 1 + \sin\theta$.



Solution: Find intersections first!!

$$3\sin\theta = 1 + \sin\theta \Rightarrow 2\sin\theta = 1$$
$$\sin\theta = \frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

So we want to take the area of the circle between $\theta = \frac{\pi}{6}$ and $\frac{5\pi}{6}$ and subtract the area of the cardoid between $\theta = \frac{\pi}{6}$ and $\frac{5\pi}{6}$

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin\theta)^2\, d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{5\pi/6} (1+\sin\theta)^2\, d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 - (1+\sin\theta)^2\, d\theta \stackrel{\text{Symmetry}}{=\!=} \int_{\pi/6}^{\pi/2} (3\sin\theta)^2 - (1+\sin\theta)^2\, d\theta$$

$$= \int_{\pi/6}^{\pi/2} \left(8\sin^2\theta - 2\sin\theta - 1\right) d\theta = \int_{\pi/6}^{\pi/2} 8\left(\frac{1}{2} - \frac{1}{2}\cos(2\theta)\right) - 2\sin\theta - 1\, d\theta$$

$$= 4\theta - 2\sin(2\theta) + 2\cos(\theta) - \theta \Big|_{\pi/6}^{\pi/2}$$

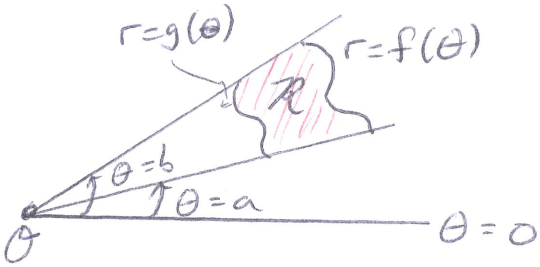
$$= \frac{3\pi}{2} - \frac{\pi}{2} + 2\frac{\sqrt{3}}{2} - 2\frac{\sqrt{3}}{2}$$

$$= \boxed{\pi}$$

In general, if you have $r = f(\theta)$, $r = g(\theta)$ between $\theta = a$ and $\theta = b$,

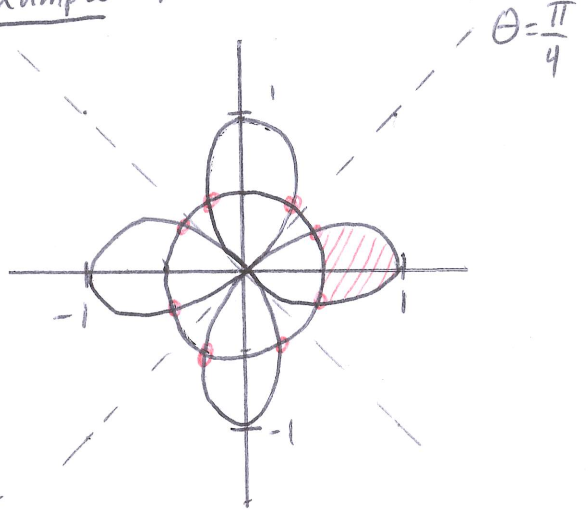
We only work between the lines $\theta = a$ and $\theta = b$!!

Not on x-axis like Cartesian!



$$\Rightarrow A = \frac{1}{2}\int_a^b \left([f(\theta)]^2 - [g(\theta)]^2\right) d\theta$$

<u>Example</u>: Find all the intersections of $r = \cos(2\theta)$ and $r = \frac{1}{2}$



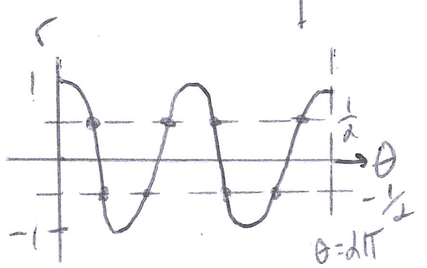
<u>Solution</u>: Note from the picture there are 8 solutions!

$\cos(2\theta) = \frac{1}{2} \Leftrightarrow \cos(u) = \frac{1}{2}$

$\Rightarrow u = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$

$\Rightarrow \boxed{\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}}$

But this is only 4! What about the others?



$\cos(2\theta) = -\frac{1}{2} \Leftrightarrow \cos(u) = -\frac{1}{2}$

$\Rightarrow u = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}$

$\Rightarrow \boxed{\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}}$

<u>Example</u>: Find the area outside $r = \frac{1}{2}$ and inside $r = \cos(2\theta)$ for the shaded petal in the above picture.

<u>Solution</u>: By symmetry, we only consider $\theta \in [0, \frac{\pi}{6}]$, then multiply by 2.

$$\Rightarrow A = 2\int_0^{\pi/6} \frac{1}{2}\left[(\cos(2\theta))^2 - \left(\frac{1}{2}\right)^2\right] d\theta = \int_0^{\pi/6} \cos^2(2\theta) - \frac{1}{4}\, d\theta$$

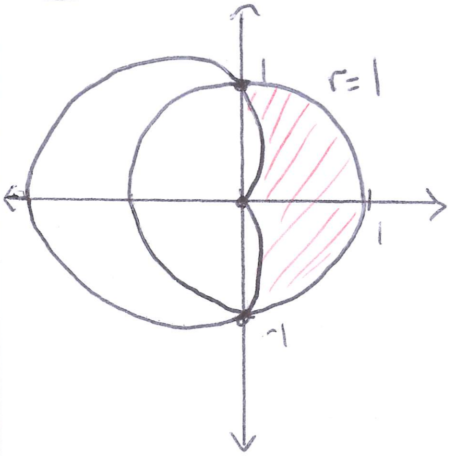
from page 1

$$= \frac{1}{2}\theta + \frac{1}{8}\sin(4\theta) - \frac{1}{4}\theta \Big|_0^{\pi/6}$$

$$= \frac{1}{4}\theta + \frac{1}{8}\sin(4\theta)\Big|_0^{\pi/6}$$

$$= \frac{\pi}{24} + \frac{1}{8}\sin\left(\frac{2\pi}{3}\right) = \boxed{\frac{\pi}{24} + \frac{\sqrt{3}}{16}}$$

Example: Find the area inside $r=1$ and outside $r=1-\cos(\theta)$.



Solution: Find the intersections

$$1-\cos(\theta)=1 \iff \cos(\theta)=0$$

$$\theta=\frac{\pi}{2}, \frac{3\pi}{2} \left(\text{or } -\frac{\pi}{2}!\right)$$

So we can go for $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, then use symmetry.

$$A=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(1^2-(1-\cos(\theta))^2\right)d\theta$$

$$=\int_0^{\pi/2}\left(1-(1-2\cos(\theta)+\cos^2(\theta))\right)d\theta$$

$$=\int_0^{\pi/2}\left(2\cos(\theta)-\frac{1}{2}-\frac{1}{2}\cos(2\theta)\right)d\theta$$

$$=2\sin(\theta)-\frac{1}{2}\theta-\frac{1}{4}\sin(2\theta)\Big|_0^{\frac{\pi}{2}}$$

$$=\boxed{2-\frac{\pi}{4}}$$

Arc Length in Polar Coordinates: $r=f(\theta)$ for $\alpha<\theta<\beta$

$x=r\cos\theta$
$y=r\sin\theta$
$\frac{dx}{d\theta}=\frac{dr}{d\theta}\cos\theta-r\sin\theta$
$\frac{dy}{d\theta}=\frac{dr}{d\theta}\sin\theta+r\cos\theta$

$$L=\int_\alpha^\beta\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta=\int_\alpha^\beta\sqrt{\left(\frac{dr}{d\theta}\right)^2\cos^2\theta+r^2\sin^2\theta-2r\frac{dr}{d\theta}\cos\theta\sin\theta+\left(\frac{dr}{d\theta}\right)^2\sin^2\theta+r^2\cos^2\theta+2r\frac{dr}{d\theta}\sin\theta\cos\theta}\,d\theta$$

$$=\int_\alpha^\beta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

Example: Find the arclength of $r=1+\sin\theta$, $0\le\theta\le 2\pi$ $\Rightarrow \frac{dr}{d\theta}=\cos\theta$

Solution: $L=\int_0^{2\pi}\sqrt{(1+\sin\theta)^2+\cos^2\theta}\,d\theta=\int_0^{2\pi}\sqrt{2+2\sin\theta+\sin^2\theta+\cos^2\theta}\,d\theta$

$$=\int_0^{2\pi}\sqrt{2+2\sin\theta}\,d\theta=\int_0^{2\pi}\sqrt{2}\sqrt{1+\sin\theta}\,d\theta$$

Complicated!

(*) $=\int_{-\pi}^{\pi}\sqrt{2+2\cos\theta}\,d\theta=\int_{-\pi}^{\pi}2\cos\left(\frac{\theta}{2}\right)d\theta$

$$=4\sin\left(\frac{\theta}{2}\right)\Big|_{-\pi}^{\pi}=4-(-4)=\boxed{8}$$

(*) Consider $r=1-\cos\theta$ $-\pi\le\theta\le\pi$ gives same plot, but rotated.

and: $\cos^2\theta=\frac{1}{2}+\frac{1}{2}\cos(2\theta)$

$4\cos^2\left(\frac{\theta}{2}\right)=2+2\cos(\theta)$

$\Rightarrow 2\cos\left(\frac{\theta}{2}\right)=\sqrt{2+2\cos\theta}$

$\cos\left(\frac{\theta}{2}\right)>0$ on $-\pi<\theta<\pi$

Use this instead!!