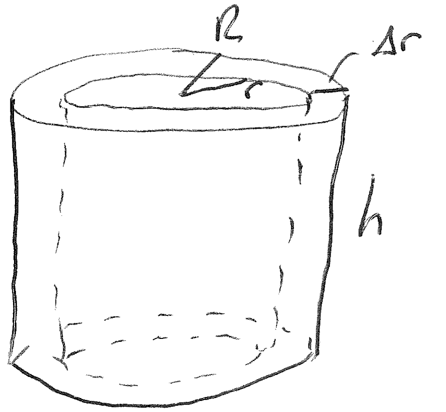
Section 6.2 - Volume Using Cylindrical Shells



The big cylinder has radius R, little cylinder has radius r. What is the volume of the cylindrical slice between them?

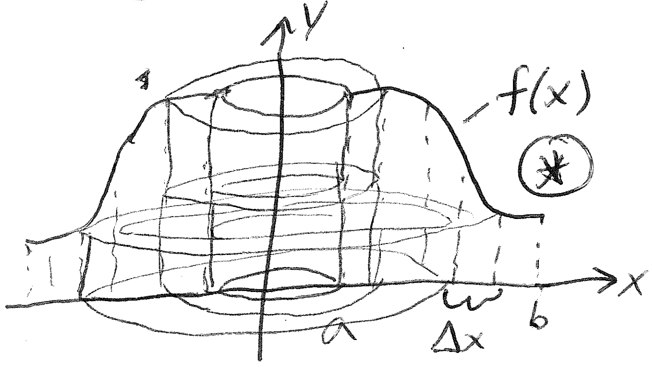
$$V = V_2 - V_1 \quad \text{where } V_2 = \text{Volume of big cyl.}, \quad V_1 = \text{Vol. of small cyl.}$$

$$\Rightarrow V = \pi R^2 h - \pi r^2 h = \pi (R^2 - r^2) h$$
$$= \pi (R+r)(R-r) h = 2\pi \frac{R+r}{2} h (R-r)$$

Letting $\Delta r = R - r$ and $\tilde{r} = \frac{R+r}{2}$

$$\Rightarrow V = 2\pi \tilde{r} h \Delta r = (\text{circumference})(\text{height})(\text{thickness})$$

So for some arbitrary function $f(x)$, we rotate about the y-axis and have shells such that for the ith shell



(*)

$$V_i = (2\pi \bar{x}_i)[f(\bar{x}_i)]\Delta x$$

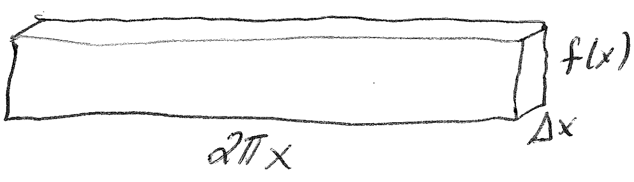
$$V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi(\bar{x}_i) f(\bar{x}_i) \Delta x$$

$$\Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi \bar{x}_i f(\bar{x}_i) \Delta x$$
$$= \int_a^b 2\pi x f(x)\, dx$$

So for a solid rotated about y-axis, the volume is

$$V = \int_a^b 2\pi x f(x) dx \quad \text{where } 0 \le a < b.$$

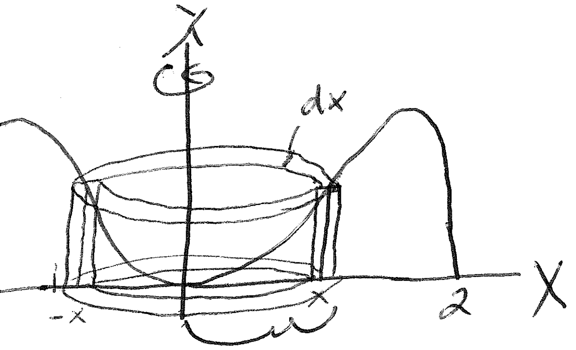
(*)



Unwrap the shell

Examples 1

① Find volume of $y=2x^2-x^3$ and $y=0$
region boundedby



$$V=\int_0^2 (2\pi x)(2x^2-x^3)\,dx$$

$$V=2\pi\int_0^2 (2x^3-x^4)\,dx$$

$$=2\pi\left(8-\frac{32}{5}\right)=\boxed{\frac{16}{5}\pi}$$

radius is x

height is function $2x^2-x^3$

$2x^2-x^3=0$

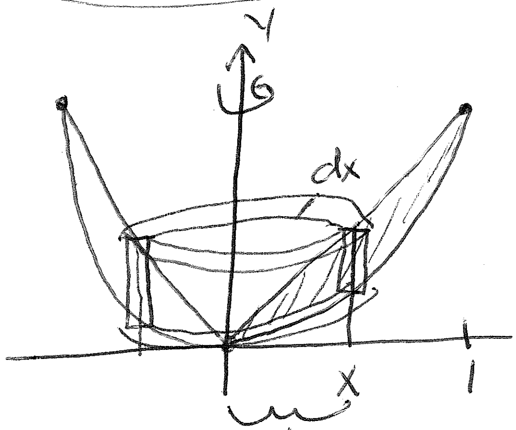
$-\cancel{2}x^2(x-2)=0$

$x=0,\ x=2$

② Find the volume of the solid obtained by rotating about y-axis the region between $y=x$ and $y=x^2$

Solution: Bounds of integration?

$$x^2=x \Rightarrow x^2-x=0$$
$$x(x-1)=0$$
$$x=0,1$$



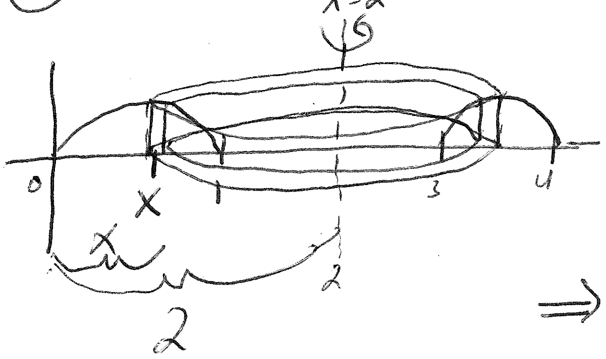
radius = x

$$V=\int_0^1 (2\pi x)(x-x^2)\,dx$$

$$=\int_0^1 2\pi(x^2-x^3)\,dx=\boxed{\frac{\pi}{6}}$$

Examples: (cont.)

(3) Find volume of the solid obtained by rotating region bdd by $y = x - x^2$ and $y = 0$ about $x = 2$ line



$\Rightarrow r = 2 - x$

Radius is $2 - x$

Circumference is $x - x^2$

$$\Rightarrow V = \int_0^1 2\pi(2-x)(x-x^2)\,dx$$

$$\Rightarrow V = 2\pi \int_0^1 (2-x)(x-x^2)\,dx$$

$$= 2\pi \int_0^1 (x^3 - 3x^2 + 2x)\,dx = \boxed{\frac{\pi}{2}}$$

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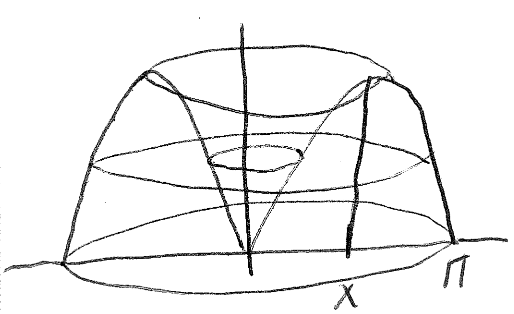
(4) Find volume of rotating $y = \sin(x)$ $0 \le x \le \pi$ about y-axis.



Radius $\Rightarrow r(x) = x$

Height $\Rightarrow h(x) = \sin(x)$

$$\Rightarrow V = 2\pi \int_0^{\pi} x\sin(x)\,dx$$

$$= 2\pi\left[-x\cos(x)\Big|_0^{\pi} + \int_0^{\pi} \cos(x)\,dx\right]$$

$$= \boxed{2\pi^2}$$