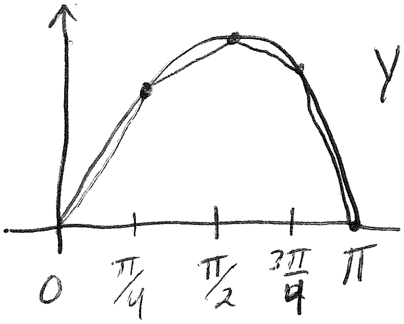
Section 6.3 - Arc Length

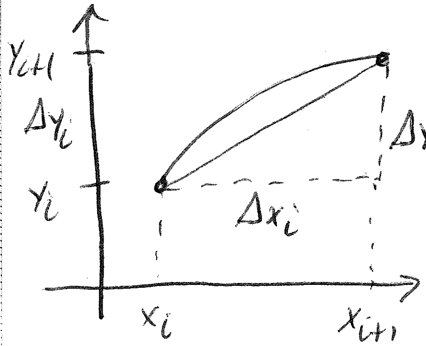
Given a curve, what is its length?



This is arc length. Imagine approximating by linear functions

Partition $[a,b]$ s.t.

$$a = x_1 < x_2 < \dots < x_n < x_{n+1} = b$$



So the linear approximation can be found by Pythagorean Thm

$$\Rightarrow L_i \approx \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$$\Rightarrow L \approx \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$c_i \in [x_i, x_{i+1}]$

$$\Rightarrow L \approx \sum_{i=1}^{n} \sqrt{1 + \frac{(\Delta y_i)^2}{(\Delta x_i)^2}}\,(\Delta x_i)$$

$f'(c_i) = \dfrac{\Delta y_i}{\Delta x_i}$

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$$\approx \sum_{i=1}^{n} \sqrt{1 + f'(c_i)^2}\,(\Delta x_i)$$

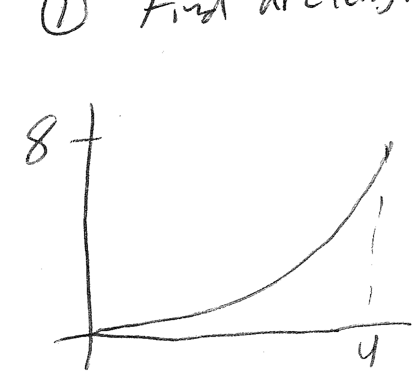
$$\Rightarrow \boxed{L = \int_a^b \sqrt{1 + f'(x)^2}\,dx}$$

Examples

① Find arclength of $f(x) = x^{3/2}$ from $x=0$ to $x=4$

$$L = \int_0^4 \sqrt{1 + \left(\tfrac{3}{2} x^{1/2}\right)^2}\,dx$$

$$= \boxed{\frac{8}{27}\left[10^{3/2} - 1\right]}$$



Examples

② Find arc length of $f(x)=\frac{1}{8}x^2-\ln x$ from $x=1$ to $x=2$

Soln: $L=\int_1^2\sqrt{1+\left(\frac{x}{4}-\frac{1}{x}\right)^2}\,dx$

$=\int_1^2\sqrt{\frac{x^2}{16}+\frac{1}{2}+\frac{1}{x^2}}\,dx$

$=\int_1^2\sqrt{\left(\frac{x}{4}+\frac{1}{x}\right)^2}\,dx$

$=\int_1^2\left(\frac{x}{4}+\frac{1}{x}\right)dx$

$=\frac{3}{8}+\ln(2)$

$\frac{x^2}{16}+\frac{1}{x^2}+\frac{1}{2}\cdot\frac{x}{x}$

$=\frac{x^2}{16}+\frac{2x}{4x}+\frac{1}{x^2}$

$=\left(\frac{x}{4}+\frac{1}{x}\right)^2$

③ Find arc length of $f(x)=x^2-\frac{1}{8}\ln(x)$ starting at $(1,1)$

Soln: $f'(x)=2x-\frac{1}{8x}$

$\Rightarrow 1+(f'(x))^2=1+\left(2x-\frac{1}{8x}\right)^2=\left(2x+\frac{1}{8x}\right)^2$

$\Rightarrow \sqrt{1+(f'(x))^2}=2x+\frac{1}{8x}$

$\Rightarrow S(x)=\int_1^x\sqrt{1+(f'(t))^2}\,dt=\int_1^x\left(2t-\frac{1}{8t}\right)dt$

$=\boxed{x^2+\frac{1}{8}\ln x-1}$