Section 6.4 - Surface Area of Revolution

The derivation for the surface area of revolution formulas is a mix between the arc length derivation from 6.3, and similar to the derivation of the shell method. If you are interested in the derivation, see the beginning of Section 6.4.

Formulas

Definition 1 If $f(x) \geq 0$ is continuously differentiable (C^1), on an interval $[a,b]$ the surface area generated by rotating $y = f(x)$ for $a \leq x \leq b$ about the $\boxed{x\text{-axis}}$ is

$$① \quad S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2}\, dx$$

or

$$② \quad S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Since $\boxed{y = f(x), \quad \frac{dy}{dx} = f'(x)}$

If the curve is $\boxed{x = g(y)}$ for $c \leq y \leq d$, and we rotate about the $\boxed{x\text{-axis}}$, we have

$$③ \quad S = \int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Definition 2 Let the same hypothesis on $f(x)$, $g(y)$ hold. If we rotate about the $\boxed{y\text{-axis}}$, we have

(a) If integrating wrt x

$$④ \quad S = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

(b) If integrating wrt y

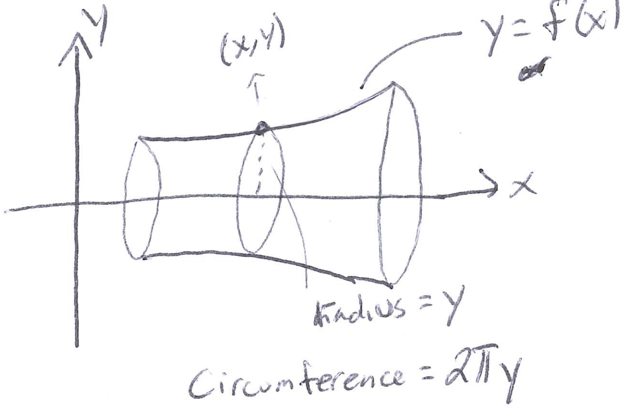
$$⑤ \quad S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

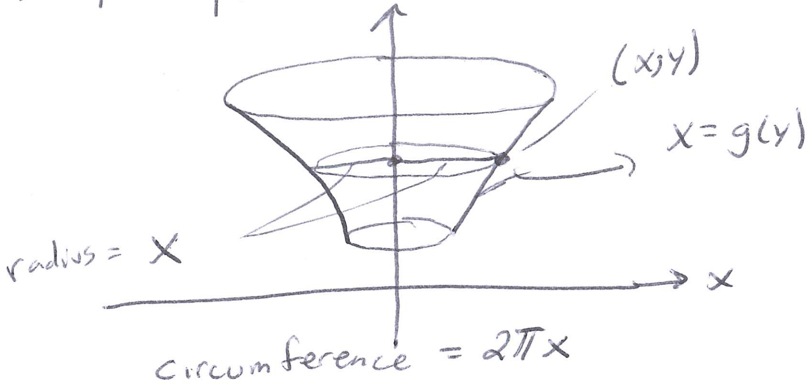
or

$$⑥ \quad S = \int_c^d 2\pi g(y) \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Since $\boxed{x = g(y), \quad \frac{dx}{dy} = g'(y)}$

The following figures help explain the situation.



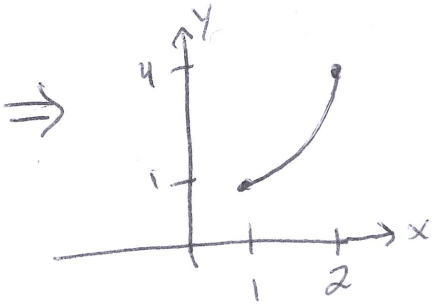


<u>Key!</u> Think of $2\pi y$ and $2\pi x$ as the circumference of a circle traced out on the surface of your region going through a point (x,y).

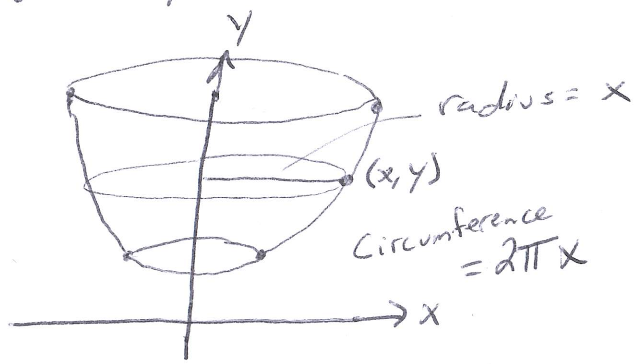
<u>Example:</u> The arc of the parabola $y=x^2$ from $(1,1)$ to $(2,4)$ is rotated about y-axis. Find the surface area.

<u>Solution:</u> To draw the picture first draw $y=x^2$ for $1 \le x \le 2$

$\Rightarrow$



Now rotate <u>this line</u> <u>only</u> around y-axis (and rotating around y-axis) $\Rightarrow$



Draw circle as in picture on right at top of page.

<u>Note:</u> Since radius $= x \Rightarrow$ use Definition 2, so choose Equation 4, 5, 6. <u>Any</u> will work, you just have to be consistent in your choice.

As we will see, both will give the right solution, and the <u>same</u> solution.

Method 1

If we choose to use 4, note radius $= x$, so it is fixed as x

$$\Rightarrow S = \int_a^b 2\pi x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \qquad \begin{matrix} a=1 \\ b=2 \end{matrix}, \quad \frac{dy}{dx} = 2x$$

$$= \int_1^2 2\pi x \sqrt{1+4x^2}\, dx \qquad u = 1+4x^2 \Rightarrow du = 8x\, dx$$

$$= 2\pi \cdot \frac{1}{8} \int_5^{17} u^{1/2}\, du = \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2} \Big|_5^{17} = \boxed{\frac{\pi}{6}\left[\underbrace{17\sqrt{17}}_{(17)^{3/2}} - \underbrace{5\sqrt{5}}_{(5)^{3/2}}\right]}$$

Which is what we got in lecture.

Now, using 5 or 6, radius is still x, but we need $x = g(y)$ to do the problem. In this case, $y = x^2 \Rightarrow x = \sqrt{y}$ in the first quadrant. So we can apply 5 or 6. If you cannot invert $y = f(x)$ to get $x = g(y)$, then you cannot use 5 or 6 !!! Now we have,

$$\Rightarrow S = \int_c^d 2\pi g(y) \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy \qquad \begin{matrix} x = g(y) = \sqrt{y} \\ \frac{dx}{dy} = g'(y) = \frac{1}{2\sqrt{y}} \end{matrix}$$

$\left.\begin{matrix} c=1 \\ d=4 \end{matrix}\right\}$ Note: we need y-bounds, see picture

$$= \int_1^4 2\pi \sqrt{y} \sqrt{1+\left(\frac{1}{2\sqrt{y}}\right)^2}\, dy$$

$$= 2\pi \int_1^4 \sqrt{y} \sqrt{1+\frac{1}{4y}}\, dy$$

$$= 2\pi \int_1^4 \sqrt{y} \sqrt{\left(\frac{1}{4y}\right)(4y+1)}\, dy$$

$$= 2\pi \int_1^4 \frac{\sqrt{y}}{\sqrt{4y}} \sqrt{4y+1}\, dy = \frac{2\pi}{\sqrt{4}} \int_1^4 \frac{\cancel{\sqrt{y}}}{\cancel{\sqrt{y}}} \sqrt{4y+1}\, dy$$

$$= \pi \int_1^4 \sqrt{4y+1}\, dy \qquad \begin{matrix} u = 4y+1 \\ du = 4\, dy \end{matrix} \quad \Rightarrow \; = \frac{\pi}{4} \int_5^{17} u^{1/2}\, du$$

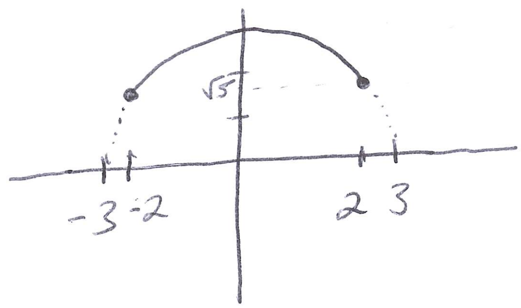
$$= \boxed{\frac{\pi}{6}\left[17\sqrt{17} - 5\sqrt{5}\right]}$$

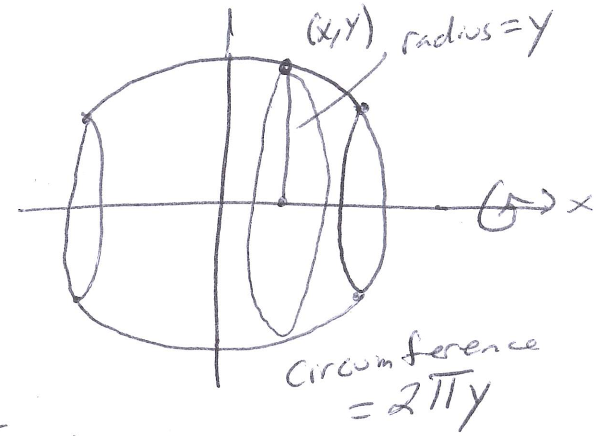
Same as above

Example: Find the surface area of the region generated by rotating $y=\sqrt{9-x^2}$ for $-2 \le x \le 2$ about the x-axis.

Solution: Draw picture first, as in previous example.

Note: $y=\sqrt{9-x^2} \Rightarrow y^2 = 9-x^2 \Rightarrow x^2+y^2=9$ circle of radius 3, center $(0,0)$. But $y=\sqrt{9-x^2}$ and $-2 \le x \le 2$ implies it is the top half of circle

$\Rightarrow$ 

$\Rightarrow$ 

So rotation about x-axis and $C=2\pi y \Rightarrow$ Use 1, 2, or 3

Using 1 is easy since $f(x)=y=\sqrt{9-x^2}$

$a=-2,\ b=2$ $\qquad f'(x)=\dfrac{dy}{dx} = -\dfrac{x}{\sqrt{9-x^2}}$ by Chain Rule.

$$\Rightarrow S = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx = 2\pi\int_{-2}^{2}\sqrt{9-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{9-x^2}}\right)^2}\,dx$$

$$= 2\pi\int_{-2}^{2}\sqrt{9-x^2}\sqrt{1+\frac{x^2}{9-x^2}}\,dx = 2\pi\int_{-2}^{2}\sqrt{9-x^2}\sqrt{\frac{9}{9-x^2}}\,dx$$

$$= 2\pi\int_{-2}^{2}\sqrt{9-x^2}\cdot\frac{3}{\sqrt{9-x^2}}\,dx = 6\pi\int_{-2}^{2}dx$$

$$= 6\pi\left(x\Big|_{-2}^{2}\right) = 6\pi(4) = \boxed{24\pi}$$