Section 6.5 – Work (and a bit from Section 6.6 – Moments)

Work done by a constant force is given by

$$W = F \cdot d.$$

The work done by a variable force $F(x)$ moving along the x-axis from a to b is

$$W = \int_a^b F(x)\,dx$$

* Units: English: ft·lb
Metric: N·m

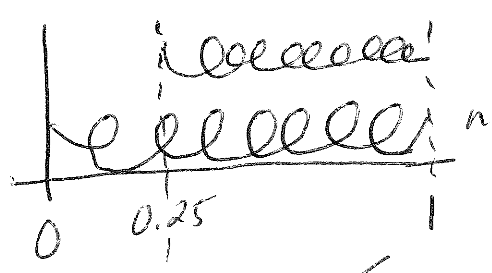
Examples

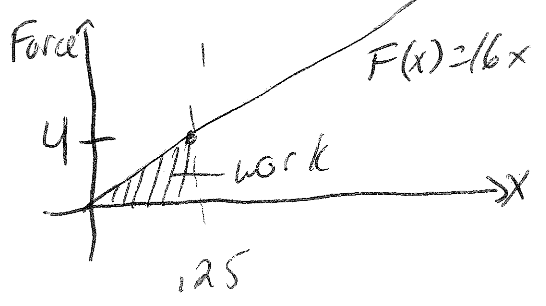
① Find the work required to compress a spring of natural length 1 ft to a length of 0.75 ft with a force constant $K = 16\ \text{lb}/\text{ft}$.

Hooke's Law: $F = Kx \Rightarrow F(x) = 16x$

$$\Rightarrow W \int_0^{.25} 16x\,dx = 8x^2\Big|_0^{1/4} = \frac{1}{2}\ \text{ft. lb.}$$

bounds of integration are the distance compressed





(2) A 5lb bucket is lifted from ground into the air by pulling a rope of 20ft at constant speed. The rope weighs 0.08 lb/ft. What is the work done to reach 20ft?

Solution: The bucket has constant weight, so the work to lift the bucket is $W_b = F_b \cdot d = 5 \cdot 20 = 100 \text{ ft lb}$

The work to lift the rope varies with x, since there is less rope as the bucket is pulled.

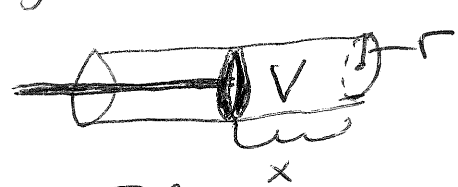
$$\Rightarrow \quad F_r(x) = (.08)(20-x) \text{ lb.}$$

$$\Rightarrow \quad W_r = \int_0^{20} (.08)(20-x)\,dx = \left[1.6x - 0.04x^2\right]\Big|_0^{20}$$

$$= 16 \text{ ft lb}$$

$$\Rightarrow \quad W = W_b + W_r = 100 + 16 = \boxed{116 \text{ ft-lb}}$$

(3) A gas expands a cylinder with radius r. The pressure at any time is a function of volume $P = P(V)$. The force exerted by the gas on the piston is $F = \pi r^2 P$. Show that the work done by the gas when volume expands is $W = \int_{V_1}^{V_2} P\,dV$



Solution: $V = \pi r^2 x$ via picture. $\Rightarrow V_1 = \pi r^2 x_1$
$V_2 = \pi r^2 x_2$

Let $V(x) = \pi r^2 x$
$\Rightarrow dV(x) = \pi r^2 dx$

$$\Rightarrow W = \int_{x_1}^{x_2} F(x)\,dx = \int_{x_1}^{x_2} \pi r^2 P(V(x))\,dx$$

$$= \boxed{\int_{V_1}^{V_2} P(V)\,dV}$$

(4) Newton's Law of Gravitation states two masses with mass m_1 and m_2 attract with force

$$F = G\frac{m_1 m_2}{r^2}$$

r = distance between
G = gravitational constant

If one body is fixed, find the work to move the other from $r=a$ to $r=b$.

$$W = \int_a^b F(r)\,dr = Gm_1m_2\int_a^b \frac{1}{r^2}\,dr = Gm_1m_2\left(-\frac{1}{r}\right)\Big|_a^b$$

$$= \boxed{Gm_1m_2\left(\frac{1}{a}-\frac{1}{b}\right)}$$