MATH 65B **PRACTICE FINAL** **May 15, 2018**

Name: ____________________________ **Score:** _______ / 100

Student ID: ____________________________

DO NOT OPEN THE EXAM UNTIL YOU ARE TOLD TO DO SO

	1	2	3	4	5	6	7	8	9	10	11	Total
✓												100
Score												
Pts. Possible	10	10	10	10	10	10	10	10	10	10	10	110

INSTRUCTIONS FOR STUDENTS

- Questions are on both sides of the paper. This is an 11 question exam.
- Students have 2 hours and 30 minutes to complete the exam.
- The test will be out of **100** points. You may attempt as many problems or parts of problems as you would like. The highest possible score is therefore **110** points.
- In the above table, the row with the ✓, is for you to keep track of the problems you are attempting/completing.
- You may complete parts of problems, as partial credit will be given based on correctness, completeness, and ideas that are leading to the correct solutions.
- **PLEASE SHOW ALL WORK**. **Any unjustified claims will receive no credit. This means you need to state which test you are using for series questions!** Clearly box your final answer.
- No notes, textbooks, phones, calculators, etc. are allowed for the exam.
- The last page of the test can be used for scratch work.

GOOD LUCK!

<u>FORMULAS:</u>

$\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$	$\dfrac{dy}{dx} = \dfrac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)}$
$\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$, for all $\lvert x\rvert < 1$	$\sin(x) = \displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$, for all $x \in \mathbb{R}$
$e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$, for all $x \in \mathbb{R}$	$\cos(x) = \displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$, for all $x \in \mathbb{R}$
$\ln(1+x) = \displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$, for $x \in (-1, 1]$	$\arctan(x) = \displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$, for $\lvert x\rvert \le 1$
$f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, for $\lvert x-a\rvert < R$	$(1+x)^m = \displaystyle\sum_{n=0}^{\infty} \binom{m}{n} x^n$, for $\lvert x\rvert < 1$

1) (10 pts.) Consider the parametric curve for $0 \le t \le \pi$, given by

$$x = \sin(t)$$
$$y = \cos(t)$$

(a) (3 pts.) Sketch the graph of the parametric curve and identify the direction for increasing values of t with an arrow.
(b) (4 pts.) Find the values of t where the the tangent line is vertical.
(c) (3 pts.) Determine for which values of t is the curve concave up.

Solution:

(a) Note that this is a special case of the problem done in lecture, with the sine and cosine reversed. We can eliminate the parameter:

$$\begin{aligned} x = \sin(t) &\quad \Rightarrow x^2 = \sin^2(t) \\ y = \cos(t) &\quad \Rightarrow y^2 = \cos^2(t) \\ \Rightarrow \quad & x^2 + y^2 = \sin^2(t) + \cos^2(t) \\ \Rightarrow \quad & x^2 + y^2 = 1 \end{aligned}$$

which is a circle of radius 1, centered at (0,0), and its direction is clockwise starting at the point (0,1). The graph is a semi-circle, since we have $0 \le t \le \pi$. The picture is given below.

(b) First compute the derivative using the correct equations:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\sin(t)}{\cos(t)} = -\tan(t).$$

Now we just solve for which t the derivative is undefined:

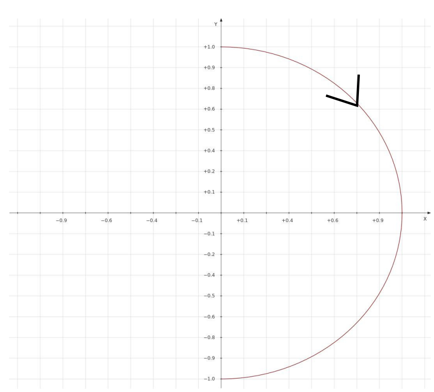
$$-\tan(t) = \text{undef} \quad \Rightarrow t = \frac{\pi}{2}$$

So the values of $t \in [0, \pi]$ is only $\frac{\pi}{2}$.

(c) First compute the second derivative using the correct equations:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{-\sec^2(t)}{\cos(t)} = -\frac{1}{\cos^3(t)}.$$

So the second derivative will be positive when $\cos(t) < 0$, which is for $t \in \left(\frac{\pi}{2}, \pi\right)$.



2) (10 pts.) Consider the polar curve for $0 \le t \le \pi$, given by

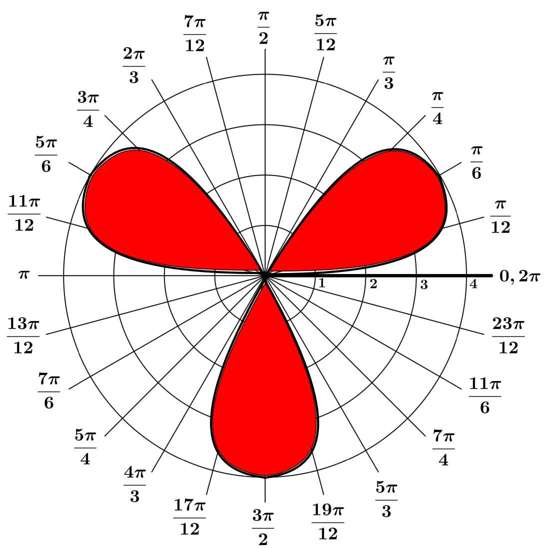
$$r = 4\sin(3\theta)$$

(a) (4 pts.) Plot the curve on the given polar grid below.

(b) (6 pts.) Find the equation of the tangent line at $\theta = \dfrac{\pi}{6}$.

Solution:

(a)



(b) Compute the first derivative using the correct formula

$$\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\dfrac{dr}{d\theta}\cos(\theta) - r\sin(\theta)} = \frac{12\cos(3\theta)\sin(\theta) + 4\sin(3\theta)\cos(\theta)}{12\cos(3\theta)\cos(\theta) - 4\sin(3\theta)\sin(\theta)}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{12\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{6}\right) + 4\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{6}\right)}{12\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{6}\right) - 4\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{6}\right)} = \frac{12(0)\left(\frac{1}{2}\right) + 4(1)\left(\frac{\sqrt{3}}{2}\right)}{12(0)\left(\frac{\sqrt{3}}{2}\right) - 4(1)\left(\frac{1}{2}\right)} = \frac{2\sqrt{3}}{-2} = -\sqrt{3}$$

Now that we have the slope, we just need the x and y coordinates. We use the polar coordinates equations

$$x = r\cos(\theta) = 4\sin(3\theta)\cos(\theta) = 4\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{6}\right) = 2\sqrt{3}$$

$$y = r\sin(\theta) = 4\sin(3\theta)\sin(\theta) = 4\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{6}\right) = 2$$

So using the point slope formula, the tangent line is simply

$$y - y_0 = \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}}(x - x_0)$$

$$y - 2 = -\sqrt{3}(x - 2\sqrt{3})$$

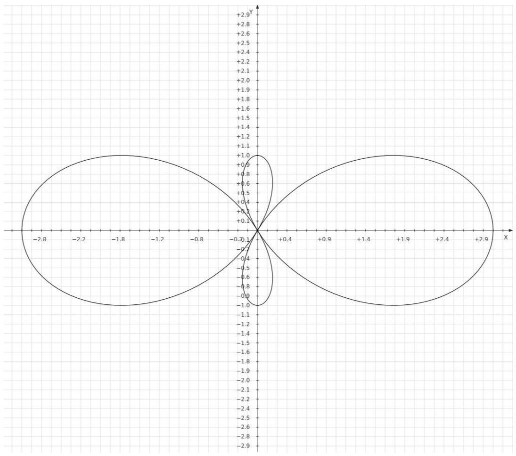
$$y - 2 = -\sqrt{3}x + 6$$

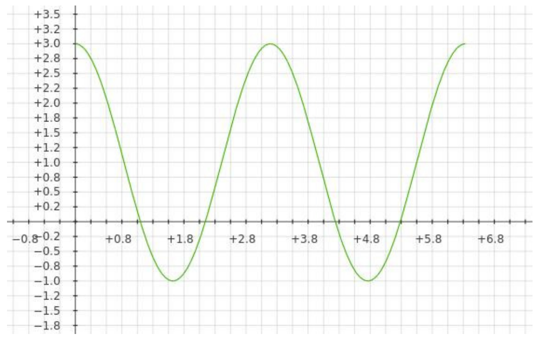
$$y = -\sqrt{3}x + 8$$

3) (10 pts.) Find the area of the region that lies inside one of the small petal and one large petal of

$$r = 1 + 2\cos(2\theta)$$

Hint: Identities that may be helpful: 1) $\sin^2(2\theta) = \frac{1}{2} - \frac{1}{2}\cos(4\theta)$, *2)* $\cos^2(2\theta) = \frac{1}{2} + \frac{1}{2}\cos(4\theta)$.





<u>Solution</u>:
From the figures above, we can deduce that we trace out one half of a large petal, then one half of a small petal for $0 \leq \theta \leq \frac{\pi}{2}$. Therefore we can integrate the function over this interval, then multiply by 2. Alternatively, we could just take the interval $0 \leq \theta \leq \pi$, as this will trace out a half large petal, the one whole small petal, then a half large petal. Either way we get

$$\begin{aligned}
A &= \int_0^{\pi} \frac{1}{2} r^2 \, d\theta \\
&= \int_0^{\pi} \frac{1}{2}(1 + 2\cos(2\theta))^2 \, d\theta \\
&= \int_0^{\pi} \frac{1}{2}(1 + 4\cos(2\theta) + 4\cos^2(2\theta)) \, d\theta \\
&= \int_0^{\pi} \frac{1}{2} + 2\cos(2\theta) + 2\cos^2(2\theta)) \, d\theta \\
&= \int_0^{\pi} \frac{1}{2} + 2\cos(2\theta) + 2\left(\frac{1}{2} + \frac{1}{2}\cos(4\theta)\right) \, d\theta \\
&= \int_0^{\pi} \frac{3}{2} + 2\cos(2\theta) + \cos(4\theta) \, d\theta \\
&= \left(\frac{3}{2}\theta + \sin(2\theta) + \frac{1}{4}\sin(4\theta)\right)\Big|_0^{\pi} \\
&= \frac{3\pi}{2}
\end{aligned}$$

4) (5 pts.) (a) Determine whether the sequence converges or diverges:

$$a_n = \left(1 + \sin\left(\frac{1}{n}\right)\right)^{\cot\left(\frac{1}{n}\right)}.$$

(5 pts.) (b) Determine whether the sequence converges or diverges:

$$a_n = \arctan(2n).$$

<u>Solution</u>:

(a) We have a function to a function, so use the exponential-logarithm trick:

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \left(1 + \sin\left(\frac{1}{n}\right)\right)^{\cot\left(\frac{1}{n}\right)} \\
&= \lim_{n\to\infty} \exp\left(\ln\left(\left(1 + \sin\left(\frac{1}{n}\right)\right)^{\cot\left(\frac{1}{n}\right)}\right)\right) = \lim_{n\to\infty} \exp\left(\frac{1}{\ln(n)} \ln\left(\frac{1}{n}\right)\right) \\
&= \lim_{n\to\infty} \exp\left(\cot\left(\frac{1}{n}\right) \ln\left(1 + \sin\left(\frac{1}{n}\right)\right)\right) = \exp\left(\lim_{n\to\infty} \frac{\ln\left(1 + \sin\left(\frac{1}{n}\right)\right)}{\tan\left(\frac{1}{n}\right)}\right) \\
&= \exp\left(\lim_{u\to 0^+} \frac{\ln\left(1 + \sin(u)\right)}{\tan(u)}\right) = \exp\left(\lim_{u\to 0^+} \frac{\cos(u)}{\sec^2(u)(1 + \sin(u))}\right) = e^1 = e
\end{aligned}$$

The limit is finite, so the sequence converges.

(b) Just apply the theorem that states that limits can be moved through continuous functions:

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \arctan(2n) \\
&= \arctan\left(2 \lim_{n\to\infty} n\right) \\
&= \arctan(\infty) \\
&= \frac{\pi}{2}
\end{aligned}$$

The limit is finite, so the sequence converges.

5) (5 pts.) (a) Determine whether the series is convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}.$$

(5 pts.) (b) Determine the rational number $\frac{p}{q}$, for $p, q \in \mathbb{Z}$ which represents the decimal expansion for $0.\overline{123} = 0.123123\ldots$, **using infinite series**.

Solution:

(a) Use the Test for Divergence:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = \frac{1}{1} = 1 \neq 0.$$

Since the limit is not equal to zero, the series diverges.

(b) Note that we can write $0.\overline{123}$ as

$$\begin{aligned}
0.\overline{123} &= 0.123 + 0.000123 + 0.000000123 + 0.000000000123 + \ldots \\
&= \frac{123}{10^3} + \frac{123}{10^6} + \frac{123}{10^9} + \frac{123}{10^{12}} + \ldots \\
&= 123 \sum_{n=1}^{\infty} \frac{1}{10^{3n}} \\
&= 123 \sum_{n=1}^{\infty} \frac{1}{10^3} \left(\frac{1}{10^3}\right)^{n-1} \\
&= 123 \cdot \frac{\frac{1}{1000}}{1 - \frac{1}{1000}} = 123 \cdot \frac{1}{1000} \cdot \frac{1000}{999} = \frac{123}{999}
\end{aligned}$$

6) (5 pts.) Determine whether the series is convergent or divergent

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^3}.$$

(b) (5 pts.) Determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right).$$

<u>Solution</u>:

(a) We can apply the integral test like in Groupwork 11. The Groupwork 11 problem is the general case, so we should already know the answer is convergent. We have the condition $a_n = f(n)$ for $f(x) = \dfrac{1}{x(\ln(x))^3}$. We must show that $f(x)$ is continuous, positive, and decreasing on $[2, \infty)$. **Continuity**: $g(x) = x$ is a line therefore continuous, and $h(x) = (\ln(x))^3$ is also continuous on $[2, \infty)$. So, the product and quotient of continuous functions is continuous, so $f(x)$ is continuous. **Positive**: Since $x > 0$ on $[2, \infty)$, and $(\ln(x))^3$ is also always positive on $[2, \infty)$, thus $f(x)$ is positive on $[2, \infty)$. **Decreasing**: To show decreasing, we can show the derivative is negative:

$$f'(x) = -\frac{3 + \ln(x)}{x^2(\ln(x))^4}.$$

If we want $f'(x) < 0$, then we require that $3 + \ln(x) > 0$, or solving for x, we have $x > e^{-3}$. So for all $x > e^{-3}$, the derivative will be negative, which is all we require to apply the Integral Test. Now we can apply the Integral Test:

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x(\ln(x))^3}\,dx &= \lim_{t\to\infty} \int_2^{t} \frac{1}{x(\ln(x))^3}\,dx \\
&= \lim_{t\to\infty} \int_{\ln(2)}^{\ln(t)} \frac{1}{u^3}\,du \qquad \text{let } u = \ln(x),\ du = \frac{1}{x}dx \\
&= \lim_{t\to\infty} \left.\frac{u^{-2}}{-2}\right|_{\ln(2)}^{\ln(t)} = \lim_{t\to\infty} \left.\frac{u^{-2}}{-2}\right|_{\ln(2)}^{\ln(t)} = \lim_{t\to\infty} \frac{\ln(t)^{-2}}{-2} - \frac{\ln(2)^{-2}}{-2} = \frac{\ln(2)^{-2}}{2} < \infty
\end{aligned}$$

Therefore, the series is convergent by the integral test.

(b) Use the Limit Comparison Test with the harmonic series. Note, we can use the limit comparison test since for $n = 1, 2, 3, \ldots$, we have that $0 \le \sin\left(\frac{1}{n}\right) \le \sin(1) \approx 0.841$, so all the terms are positive. We also use the trick from the previous quiz:

$$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{t\to 0^+} \frac{\sin(t)}{t} = 1$$

Therefore, since the limit is positive and finite, and the harmonic series diverges, the original series diverges by limit comparison test.

7) (5 pts.) (a) Determine whether the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}.$$

(5 pts.) (b) Determine whether the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=0}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2}.$$

<u>Solution</u>:

(a) Since we have factorials, the Ratio Test is the natural candidate. Applying the Ratio Test, we have

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(n+1)!}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n!}\right| \\
&= \lim_{n\to\infty} \frac{(n+1)\cdot n!}{n!} \cdot \frac{e^{n^2}}{e^{n^2+2n+1}} \\
&= \lim_{n\to\infty} \frac{n+1}{1} \cdot \frac{1}{e^{2n+1}} \\
&= \lim_{n\to\infty} \frac{n+1}{e^{2n+1}} \\
&= 0
\end{aligned}$$

Since the limit is less than 1, we have that the series converges absolutely by the Ratio Test.

(b) Since we have powers of n, the Root Test should be applied. Applying the Root Test, we get

$$\begin{aligned}
\lim_{n\to\infty} (|a_n|)^{\frac{1}{n}} &= \lim_{n\to\infty} \left(\left(1 + \frac{1}{n}\right)^{n^2}\right)^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{n} \\
&= e
\end{aligned}$$

Since the limit is greater than 1, we have that the series diverges by the Root Test.

8) (10 pts.) Determine whether the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=2}^{\infty}(-1)^n \frac{n^3}{n^4+1}.$$

Solution:

(1) Check absolute convergence first: use limit comparison with harmonic series. Take absolute value:

$$\sum_{n=2}^{\infty}\left|(-1)^n \frac{n^3}{n^4+1}\right| = \sum_{n=2}^{\infty}\frac{n^3}{n^4+1}.$$

Now do the comparison

$$\begin{aligned}
\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{n^3}{n^4+1}\cdot\frac{n}{1} \\
&= \lim_{n\to\infty}\frac{n^4}{n^4+1}\cdot\frac{\frac{1}{n^4}}{\frac{1}{n^4}} \\
&= \lim_{n\to\infty}\frac{1}{1+\frac{1}{n^4}} \\
&= 1
\end{aligned}$$

Since the limit is a finite positive number, and the harmonic series is divergent, the above series *does not* converge absolutely.

(2) Check Alternating Series Test:

(a) $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n^3}{n^4+1} = 0$

(b) Now we must show that b_n is decreasing. Take derivative on $f(x) = \dfrac{x^3}{x^4+1}$:

$$f'(x) = -\frac{x^2(x^4-3)}{(x^4+1)^2} < 0 \qquad \text{for } x^4-3>0 \;\Rightarrow\; x > \sqrt[4]{3}$$

So the sequence converges by Alternating Series Test. Since the series did not converge absolutely, the series is **conditionally convergent**.

9) (10 pts.) Find the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=0}^{\infty}(-1)^n\frac{(x-3)^n}{2n+1}$$

Solution:

Use the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|(-1)^{n+1}\frac{(x-3)^{n+1}}{2(n+1)+1}\cdot\frac{2n+1}{(-1)^n(x-3)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right|\cdot\frac{2n+1}{2n+3} \\
&= |x-3|\lim_{n\to\infty}\frac{2n+1}{2n+3} = |x-3|<1 \qquad \Rightarrow\ R=1.
\end{aligned}$$

Now, we must find the interval of convergence and check the endpoints. Solving the inequality gives

$$|x-3|<1 \qquad \Rightarrow\ 2<x<4 \qquad \Rightarrow\ I=(2,4)$$

Checking the endpoints gives

$$x=2 \qquad \Rightarrow \sum_{n=0}^{\infty}\frac{(-1)^n(-1)^n}{2n+1}=\sum_{n=0}^{\infty}\frac{1}{2n+1} \qquad \Rightarrow \text{ divergent by LCT with harmonic series}$$

$$x=4 \qquad \Rightarrow \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}= \qquad \Rightarrow \text{ convergent by Alternating Series Test}$$

Therefore the interval of convergence is $I=(2,4]$.

10) (10 pts.) Compute the first 4 non-zero terms of the Taylor series centered at $a = 3$ for the following function using the definition of a Taylor series.

$$f(x) = e^x$$

Solution:

To do this problem by the definition, recall the general formula for the Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

so we only need to compute the coefficients which are generated using the derivatives of $f(x)$. So we get

$$\begin{aligned}
f^{(0)}(x) = e^x \quad &\Rightarrow \quad f^{(0)}(3) = e^3 \\
f^{(1)}(x) = e^x \quad &\Rightarrow \quad f^{(1)}(3) = e^3 \\
f^{(2)}(x) = e^x \quad &\Rightarrow \quad f^{(2)}(3) = e^3 \\
f^{(3)}(x) = e^x \quad &\Rightarrow \quad f^{(3)}(3) = e^3
\end{aligned}$$

Now we can compute the whole coefficient by dividing by the $n!$ to get

$$\begin{aligned}
\frac{f^{(0)}(3)}{0!} &= e^3 \\
\frac{f^{(1)}(3)}{1!} &= e^3 \\
\frac{f^{(2)}(3)}{2!} &= \frac{e^3}{2} \\
\frac{f^{(3)}(3)}{3!} &= \frac{e^3}{6}
\end{aligned}$$

Therefore, the Taylor series approximation of $f(x)$ is

$$f(x) = e^x \approx \sum_{n=0}^{3} \frac{f^{(n)}(a)}{n!}(x-a)^n = e^3\left(1 + (x-3) + \frac{1}{2}(x-3)^2 + \frac{1}{6}(x-3)^3\right)$$

11) (5 pts.) (a) Compute the following integral using Taylor series, and leave the answer as a Taylor series.

$$\int \cos(x^2)\ dx$$

(5 pts.) (b) Find the Taylor series centered at $a = 0$ for the function and write as a single series.

$$f(x) = e^{-x^2} + \cos(x)$$

<u>Solution</u>:

(a) We know from the table on the front, the Taylor series for cosine:

$$\begin{aligned}
\cos(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\
\cos(x^2) &= \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!} \\
\Rightarrow \quad \int \cos(x^2)\ dx &= \int \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!} \right)\ dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\int x^{4n}\ dx \right) \\
&= C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{x^{4n+1}}{4n+1}
\end{aligned}$$

(b) Use the formula for the Taylor Series centered at 0 from the front page of the test:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \Rightarrow \quad e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

Compute the numerator first by writing out the terms for the above series

$$\begin{aligned}
e^{-x^2} + \cos(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\
&= \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{n!} + \frac{1}{(2n)!} \right) x^{2n}
\end{aligned}$$

THIS PAGE IS LEFT BLANK FOR ANY SCRATCH WORK

END OF TEST