MATH 65B PRACTICE MIDTERM March 8, 2018

Name: KEY Score: ________ / 100

Student ID: ______________________

DO NOT OPEN THE EXAM UNTIL YOU ARE TOLD TO DO SO

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Total |
|---|---|---|---|---|---|---|---|---|---|---|
| ✓ | | | | | | | | | | 70 |
| Score | | | | | | | | | | |
| Pts. Possible | 10 | 5 | 10 | 5 | 10 | 10 | 10 | 10 | 5 | 75 |

## INSTRUCTIONS FOR STUDENTS

- Questions are on both sides of the paper. This is an 9 question exam.
- Students have 1 hour and 50 minutes to complete the exam.
- The test will be out of **70 points**. The highest possible score will be **75 points**. You can attempt as many of the questions as you wish, but keep in mind you are trying to get to the **70 points**.
- In the above table, the row with the ✓, is for you to keep track of the problems you are attempting/completing.
- Higher point problems are harder, thus they are weighted more. In order to do well, you will have to attempt some of the more difficult problems.
- You may complete parts of problems, as partial credit will be given based on correctness, completeness, and ideas that are leading to the correct solutions.
- **PLEASE SHOW ALL WORK**. Any unjustified claims will receive no credit. Clearly box your final answer.
- No notes, textbooks, phones, calculators, etc. are allowed for the exam.
- The back of the test can be used for scratch work.

GOOD LUCK!

**FORMULAS:**

| Useful Formulas | Useful Formulas |
|---|---|
| $\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}} \quad \lvert x\rvert < 1$ | $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + C$ |
| $\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}} \quad \lvert x\rvert < 1$ | $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$ |
| $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$ | $\int \frac{dx}{x\sqrt{a^2-x^2}} = \frac{1}{a}\operatorname{arcsec}\left\lvert\frac{x}{a}\right\rvert + C$ |
| $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ | $\cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta))$ |
| $\sin^2(x) + \cos^2(x) = 1$ | $\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))$ |

1) (10 pts.) Compute the following indefinite integral.

$$\int x^2 \sin(2x)\, dx$$

Let $u = x^2$ $\quad dv = \sin(2x)\,dx$ $\qquad \int u\,dv = uv - \int v\,du$

$du = 2x\,dx$ $\quad v = -\frac{1}{2}\cos(2x)$

$$\Rightarrow \int x^2 \sin(2x)\,dx = -\frac{1}{2}x^2\cos(2x) + \frac{1}{\cancel{2}}\cdot\cancel{2}\underbrace{\int x\cos(2x)\,dx}$$

Let $u = x$ $\quad dv = \cos(2x)\,dx$

$du = dx$ $\quad v = \frac{1}{2}\sin(2x)$

$$\Rightarrow \int x^2 \sin(2x)\,dx = -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) - \frac{1}{2}\int \sin(2x)\,dx$$

$$= \boxed{-\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C}$$

$\sin^2(x) = \frac{1}{2}(1-\cos(2x))$

$\cos^2(x) = \frac{1}{2}(1+\cos(2x))$

2) (5 pts.) Compute the following indefinite integral.

both powers even

$$\int \sin^2(x)\cos^4(x)\,dx$$

Note: $\sin^2(x)\cos^4(x) = \frac{1}{2}(1-\cos(2x))\left(\frac{1}{2}(1+\cos(2x))\right)^2$

$= \frac{1}{8}\left(1+\cos(2x)-\cos^2(2x)-\cos^3(2x)\right)$

$= \frac{1}{8}\left(1+\cos(2x)-\frac{1}{2}(1+\cos(4x))-\cos^3(2x)\right)$

$\Rightarrow \int \sin^2(x)\cos^4(x)\,dx = \frac{1}{8}\int\left(1+\cos(2x)-\frac{1}{2}-\frac{1}{2}\cos(4x)-\cos^3(2x)\right)dx$

$= \frac{1}{8}\int\frac{1}{2}\,dx + \frac{1}{8}\int\cos(2x)\,dx - \frac{1}{16}\int\cos(4x)\,dx - \frac{1}{8}\int\cos^3(2x)\,dx$

$= \frac{1}{16}x + \frac{1}{16}\sin(2x) - \frac{1}{64}\sin(4x) - \frac{1}{8}\int\cos^3(2x)\,dx$

For $\int\cos^3(2x)\,dx = \int\cos^2(2x)\cos(2x)\,dx = \int(1-\sin^2(2x))\cos(2x)\,dx$ $\quad u = \sin(2x)$, $du = \cos(2x)\,dx$

$= \frac{1}{2}\int(1-u^2)\,du$

$= \frac{1}{2}\left(u-\frac{u^3}{3}\right) = \frac{1}{2}\left(\sin(2x)-\frac{\sin^3(2x)}{3}\right)$

$= \frac{1}{2}\left(\sin(2x)-\frac{\sin^3(2x)}{3}\right)$

$$\Rightarrow \boxed{\int \sin^2(x)\cos^4(x)\,dx = \frac{1}{16}x + \frac{1}{16}\sin(2x) - \frac{1}{64}\sin(4x) - \frac{1}{16}\left(\sin(2x)-\frac{\sin^3(2x)}{3}\right) + C}$$

3) (10 pts.) Compute the following indefinite integral.

$$\int \frac{\sqrt{1+x^2}}{x}\, dx$$

Use $x = \tan\theta$ , $a = 1$

$dx = \sec^2\theta \, d\theta$

$$\Rightarrow \int \frac{\sqrt{1+x^2}}{x}\, dx = \int \frac{\sqrt{1+\tan^2\theta}}{\tan\theta} \sec^2\theta \, d\theta$$

$$= \int \frac{\sec\theta}{\tan\theta} \sec^2\theta \, d\theta$$

$$= \int \frac{\sec\theta}{\tan\theta}(1+\tan^2\theta)\, d\theta$$

$$= \int (\csc\theta + \sec\theta\tan\theta)\, d\theta$$

$$= \ln|\csc\theta - \cot\theta| + \sec\theta + C$$

$$= \boxed{\ln\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right| + \sqrt{1+x^2} + C}$$

$\int \csc\theta \, d\theta =$

$$= \int \csc\theta \, \frac{\csc\theta - \cot\theta}{\csc\theta - \cot\theta}\, d\theta$$

$u = \csc\theta - \cot\theta$

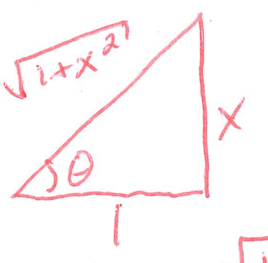
$du = (\csc^2\theta - \csc\theta\cot\theta)\, d\theta$

$$\Rightarrow \int \csc\theta \, d\theta = \int \frac{1}{u}\, du$$

$= \ln|u| + C$

$= \ln|\csc\theta - \cot\theta| + C$

$x = \tan\theta$




$\csc\theta = \dfrac{\sqrt{1+x^2}}{x}$

$\cot\theta = \dfrac{1}{x}$

$\sec\theta = \sqrt{1+x^2}$

4) (5 pts.) Compute the following indefinite integral.

$$\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx$$

Note: $\dfrac{5x^2+3x-2}{x^3+2x^2} = \dfrac{5x^2+3x-2}{x^2(x+2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+2}$

$\Rightarrow 5x^2+3x-2 = Ax(x+2) + B(x+2) + Cx^2$

Let $x=0 \Rightarrow -2 = 2B \Rightarrow B=-1$

Let $x=-2 \Rightarrow 12 = 4C \Rightarrow C=3$

Then, $5x+3x-2 = Ax(x+2) - (x+2) + 3x^2$

$\Rightarrow 2x^2+4x = Ax^2+2Ax \Rightarrow A=2$

$\Rightarrow \dfrac{5x^2+3x-2}{x^3+2x^2} = \dfrac{2}{x} + \dfrac{-1}{x^2} + \dfrac{3}{x+2}$

$\Rightarrow \displaystyle\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx = 2\int \frac{1}{x}\,dx - \int \frac{1}{x^2}\,dx + 3\int \frac{1}{x+2}\,dx$

$= \boxed{2\ln|x| + \dfrac{1}{x} + 3\ln|x+2| + C}$

5) (10 pts.) Determine whether the integral is convergent or divergent.

$$\int_{-1}^{0} \frac{e^{1/x}}{x^3}\,dx$$

$$\int_{-1}^{0} \frac{e^{1/x}}{x^3}\,dx = \lim_{t\to 0^-} \int_{-1}^{t} \frac{e^{1/x}}{x^3}\,dx = \lim_{t\to 0^-} \int_{-1}^{t} \frac{1}{x} e^{1/x} \cdot \frac{1}{x^2}\,dx$$

Let $u = \frac{1}{x} \Rightarrow du = -\frac{1}{x^2}\,dx \Rightarrow -du = \frac{1}{x^2}\,dx$

$$= -\lim_{t\to 0^-} \int_{-1}^{1/t} u e^{u}\,du$$ integration by parts

$$= -\lim_{t\to 0^-} e^{u}(u-1)\Big|_{-1}^{1/t}$$

$$= \lim_{t\to 0^-} \left(-2e^{-1} - \left(\tfrac{1}{t}-1\right)e^{1/t}\right)$$

If you let $s = \frac{1}{t}$

$\Rightarrow t \to 0^-$

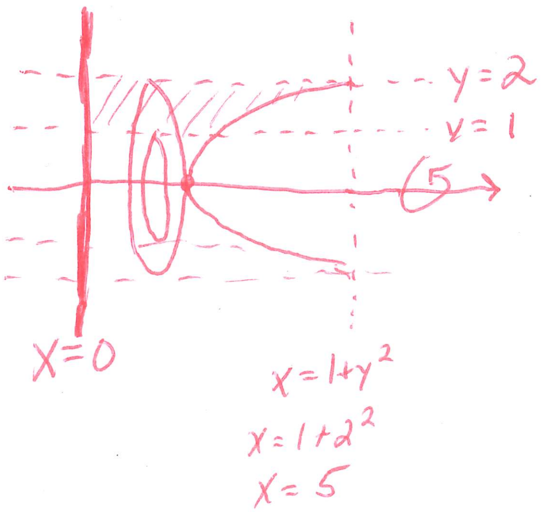
then $s \to -\infty$

$$= \frac{-2}{e} - \lim_{s\to -\infty} \frac{s-1}{e^{-s}}$$

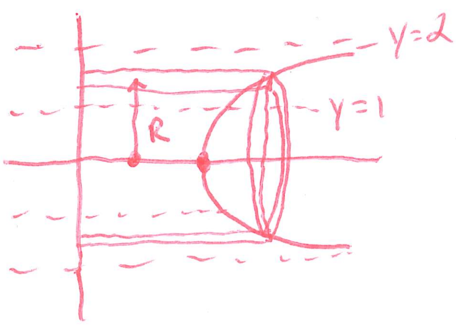
$$\underset{=}{\text{L'H}} \frac{-2}{e} - \lim_{s\to-\infty} \frac{1}{-e^{-s}}$$

$$= \boxed{-\frac{2}{e}}$$ convergent

6) (10 pts.) Find the volume of the region rotated around the $x$-axis, and bounded by $x = 1 + y^2$, $x = 0$, $y = 1$, and $y = 2$.




Slices are thickness of dx, going from x=0 to x=5 for washers. Note x=1+y², we would need y=f(x) to use. Complicated; use shells instead




$$V = 2\pi \int_a^b y\, f(y)\, dy \quad \text{where } f(y) = 1 + y^2$$

$$\Rightarrow V = 2\pi \int_1^2 y(1+y^2)\, dy$$

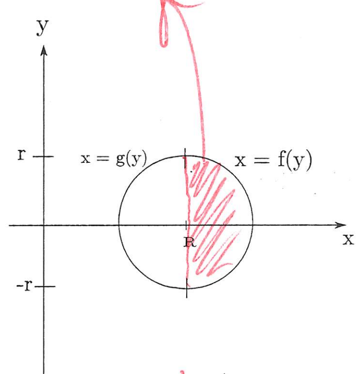
$$\Rightarrow V = 2\pi \int_1^2 y + y^3\, dy$$

$$\Rightarrow V = 2\pi \left(\frac{y^2}{2} + \frac{y^4}{4}\right)\Bigg|_1^2$$

$$\Rightarrow V = 2\pi \cdot \frac{21}{4} \Rightarrow \boxed{V = \frac{21}{2}\pi}$$

7) (10 pts.) The following question is designed to walk you through how to find the volume of the torus, the doughnut shaped solid pictured at the bottom of the page.

a) (1 pts.) Write the equation of a circle with center at $(R, 0)$ and radius $r$.
b) (2 pts.) Solve the equation of the circle you found in part (a) for $x$. Now, using the result, write out the functions $g(y)$ and $f(y)$ that are given in the diagram below.
c) (4 pts.) Integrating with respect to $y$, set up the integral for the volume, rotating around the $y$-axis. We are really doing washers from scratch. We are finding the area of the large outside disk and small inner disk. (***Hint:*** *Since we are integrating with respect to $y$, the formula is no longer $top - bottom$, but $right - left$.*)
d) (3 pts.) Do the integration and find the volume, using the answer from part (c). (***Hint:*** *What area is* $\int_{-r}^{r} \sqrt{r^2 - y^2}\, dy$ in the diagram below? Doing the integral directly is possible, but it is a trig-sub.)




a) $(x-R)^2 + y^2 = r^2$

b) $\Rightarrow x = R \pm \sqrt{r^2 - y^2}$

$\Rightarrow x = f(y) = R + \sqrt{r^2 - y^2}$

$x = g(y) = R - \sqrt{r^2 - y^2}$

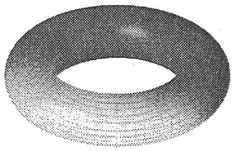
c) $V = \pi \int_a^b \left[ f(y)^2 - g(y)^2 \right] dy = \pi \int_{-r}^{r} \left[ \left(R + \sqrt{r^2 - y^2}\right)^2 - \left(R - \sqrt{r^2 - y^2}\right)^2 \right] dy$

$= \pi \int_{-r}^{r} \left[ \left(\cancel{R^2} + 2R\sqrt{r^2 - y^2} + \cancel{r^2 - y^2}\right) - \left(\cancel{R^2} - 2R\sqrt{r^2 - y^2} + \cancel{r^2 - y^2}\right) \right] dy$

$= \pi \int 4R\sqrt{r^2 - y^2}\, dy = 4\pi R \int_{-r}^{r} \sqrt{r^2 - y^2}\, dy$

d) $\int_{-r}^{r} \sqrt{r^2 - y^2}\, dy$ is area of half a circle with radius $r$

$\therefore \frac{1}{2}\pi r^2 \Rightarrow V = 4\pi R \int_{-r}^{r} \sqrt{r^2 - y^2}\, dy$



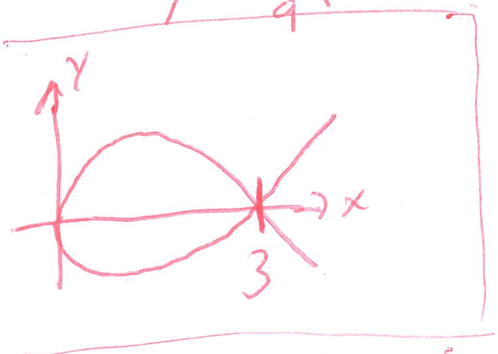
$\Rightarrow \boxed{V = 2\pi^2 r^2 R}$

8) (10 pts.) Find the area of the surface generated by rotating the loop of the curve $9y^2 = x(3-x)^2$ about the $x$-axis.

Recall the formula $S = \int_a^b 2\pi y \, ds = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$

Find $\frac{dy}{dx}$ by implicit differentiation

$$y^2 = \frac{x}{9}(3-x)^2 \xrightarrow{\frac{d}{dx}} 2y\frac{dy}{dx} = \frac{1}{9}\left(-2x(3-x) + (3-x)^2\right)$$




$$\Rightarrow \frac{dy}{dx} = \frac{(3-x)[-2x+3-x]}{18y} = \frac{(3-x)(-3x+3)}{18y}$$

$9y^2 = x(3-x)^2$

$y = \frac{\sqrt{x}(3-x)}{3}$

or $y^2 = \frac{x(3-x)^2}{9}$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{(3-x)^2(3-3x)^2}{324y^2} = \frac{(3-x)^2(3-3x)^2}{324} \cdot \frac{9}{x(3-x)^2}$$

$$= \frac{(3-3x)^2}{36x} \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = \frac{(3+3x)^2}{36x} \text{ (combine terms)}$$

$$\Rightarrow S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

bounds are 0 to 3 by picture above $a=0, b=3$

$$= 2\pi \int_0^3 \frac{\sqrt{x}(3-x)}{3} \sqrt{\frac{(3+3x)^2}{36x}} \, dx$$

$$= \frac{2\pi}{18} \int_0^3 (3-x)(3+3x) \, dx = \frac{\pi}{9} \int_0^3 (9+6x-3x^2) \, dx$$

$$= \frac{\pi}{9}\left(9x + 3x^2 - x^3\right)\Big|_0^3$$

$$= \boxed{3\pi}$$

9) (5 pts.) A particle is moved along the $x$-axis by a force that measures $\frac{10}{(1+x)^2}$ pounds at a point $x$ feet from the origin. Find the work done in moving the particle from the origin to a distance of 9 ft.

$W = \int_a^b F(x)\,dx$ $\quad F(x) = \frac{10}{(1+x)^2}$ lbs at point $x$ ft from origin

moving from $a=0$ to $b=9$

$$\Rightarrow W = \int_0^9 \frac{10}{(1+x)^2}\,dx = 10\int_1^{10} \frac{1}{u^2}\,du \qquad u = 1+x, \quad du = dx$$

$$= 10\left(-\frac{1}{u}\Big|_1^{10}\right) = 10\left(-\frac{1}{10}+1\right)$$

$$= \boxed{9 \text{ ft} \cdot \text{lbs}}$$

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END OF TEST