Math 009C - §9.5, Lecture Problem # 1

Worked Solution

1. Let $r = 1 + \sin(\theta)$ for $0 \le \theta \le 2\pi$. (a) Find the tangent line at the value of $\theta = \dfrac{\pi}{4}$. (b) Find the values of θ where there is a horizontal or vertical tangent line.

Solution: (a) First we compute the derivative and plug in $\theta = \pi/4$ to find the slope.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)} \\
&= \frac{\cos(\theta)\sin(\theta) + [1+\sin(\theta)]\cos(\theta)}{\cos^2(\theta) - [1+\sin(\theta)]\sin(\theta)} \\
&= \frac{2\cos(\theta)\sin(\theta) + \cos(\theta)}{\cos^2(\theta) - \sin(\theta) - \sin^2(\theta)} \\
\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}} &= \frac{2\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}}{\left(\frac{\sqrt{2}}{2}\right)^2 - \frac{\sqrt{2}}{2} - \left(\frac{\sqrt{2}}{2}\right)^2} = \frac{1 + \frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = -\frac{2}{\sqrt{2}} - 1 = -\sqrt{2} - 1
\end{aligned}$$

Now we find the values of the x and y coordinates of the points using the original polar equation.

$$y - y_0 = \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}}(x - x_0)$$

$$\Rightarrow\; y_0 = r\sin(\theta) = [1+\sin(\theta)]\sin(\theta) = \sin(\theta) + \sin^2(\theta) = \frac{\sqrt{2}}{2} + \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}\left(\sqrt{2}+1\right)$$

$$\Rightarrow\; x_0 = r\cos(\theta) = [1+\sin(\theta)]\cos(\theta) = \cos(\theta) + \cos(\theta)\sin(\theta) = \frac{1}{2}\left(\sqrt{2}+1\right)$$

$$\Rightarrow\; \left(y - \left(\frac{1}{2}\left(\sqrt{2}+1\right)\right)\right) = \left(-\sqrt{2}-1\right)\left(x - \left(\frac{1}{2}\left(\sqrt{2}+1\right)\right)\right)$$

(b) For the horizontal tangent lines, set the numerator of the derivative equal to zero, and solve for θ.

$$\begin{aligned}
\frac{dy}{d\theta} &= 0 \\
2\cos(\theta)\sin(\theta) + \cos(\theta) &= 0 \\
\cos(\theta)(2\sin(\theta)+1) &= 0
\end{aligned}$$

$$\cos(\theta) = 0 \quad \text{and} \quad 2\sin(\theta) + 1 = 0 \text{ or } \sin(\theta) = -\frac{1}{2}$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{and} \quad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

For the vertical tangent lines, set the denominator of the derivative equal to zero, and solve for θ. We also need to use the identity $\cos^2(t) = 1 - \sin^2(t)$.

$$\begin{aligned}
\frac{dx}{d\theta} &= 0 \\
\cos^2(\theta) - \sin(\theta) - \sin^2(\theta) &= 0 \\
1 - \sin^2(t) - \sin(\theta) - \sin^2(\theta) &= 0 \\
2\sin^2(\theta) + \sin(\theta) - 1 &= 0 \\
(2\sin(\theta) - 1)(\sin(\theta) + 1) &= 0 \\
\sin(\theta) + 1 = 0 \quad \text{and} \quad & 2\sin(\theta) - 1 = 0 \\
\sin(\theta) = -1 \quad \text{and} \quad & \sin(\theta) = \frac{1}{2} \\
\theta = \frac{3\pi}{2} \quad \text{and} \quad & \theta = \frac{\pi}{6}, \frac{5\pi}{6}
\end{aligned}$$

Note that at $\theta = \frac{3\pi}{2}$, we have the indeterminate form $\frac{0}{0}$. Thus we have to take the limit. Using L'Hopital's Rule, and computing the right and left limits:

$$\begin{aligned}
\lim_{\theta \to \left(\frac{3\pi}{2}\right)^+} -\frac{2\cos(\theta)\sin(\theta) + \cos(\theta)}{2\sin^2(\theta) + \sin(\theta) - 1} &= \lim_{\theta \to \left(\frac{3\pi}{2}\right)^+} -\frac{-2\sin^2(\theta) + 2\cos^2(\theta) - \sin(\theta)}{4\sin(\theta)\cos(\theta) + \cos(\theta)} \\
&= \frac{-2(-1)^2 + 2(0^-)^2 - (-1)}{4(-1)(0^-) + 0^-} = \frac{-1}{0^+} = -\infty
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\theta \to \left(\frac{3\pi}{2}\right)^-} \frac{2\cos(\theta)\sin(\theta) + \cos(\theta)}{2\sin^2(\theta) + \sin(\theta) - 1} &= \lim_{\theta \to \left(\frac{3\pi}{2}\right)^-} \frac{-2\sin^2(\theta) + 2\cos^2(\theta) - \sin(\theta)}{4\sin(\theta)\cos(\theta) + \cos(\theta)} \\
&= \frac{-2(-1)^2 + 2(0^+)^2 - (-1)}{4(-1)(0^+) + 0^+} = \frac{-1}{0^-} = \infty
\end{aligned}$$

The 0^+ and 0^- denote that the value is a very small positive or negative number, respectively, that is approaching zero. This allows us to get the signs on the infinities. Therefore, at $\theta = \frac{3\pi}{2}$, there is neither a vertical nor horizontal tangent line, because **the limit does not exists**, because the limit is two difference values from the left and right hand sides.
