MATH 25 FINAL October 5, 2016

Name: KEY

Score: ________ / 100

Student ID: ______________________________

DO NOT OPEN THE EXAM UNTIL YOU ARE TOLD TO DO SO

	1	2	3	4	5	6	7	8	9	10	Total
✓											27
Score											
Pts. Possible	3	3	3	3	3	3	3	3	3	3	29

INSTRUCTIONS FOR STUDENTS

- Questions are on both sides of the paper. This is an 10 question exam.
- Students have 2 hours and 15 minutes to complete the exam.
- The test will be out of **27 points**. The highest possible score will be **29 points**. You must complete 9 problems for credit (3 points each, 27 points total). If you wish, you can attempt a 10^{th} problem for extra credit. That question will be out of 2 points, for a maximum of 29 possible points.
- In the above table, the row with the ✓ should be marked for the 9 questions you want graded. Mark **EC** for the extra credit problem.
- You may complete parts of problems, as partial credit will be given based on correctness, completeness, and ideas that are leading to the correct solutions.
- **PLEASE SHOW ALL WORK**. Any unjustified claims will receive no credit. Clearly box your final answer.
- No notes, textbooks, phones, calculators, etc. are allowed for the exam.
- The back of the test can be used for scratch work.

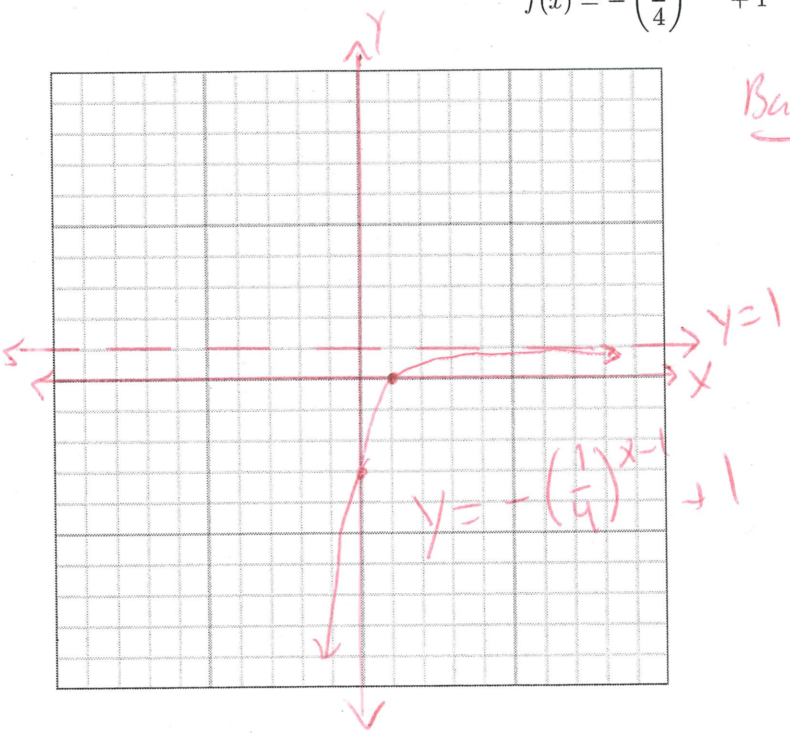
GOOD LUCK!

FORMULAS:

Name	Formula
Foci for ellipse	$c^2 = a^2 - b^2, a > b$
Foci for hyperbola	$c^2 = a^2 + b^2$
n^{th} term of Arithmetic Series	$a_n = a_1 + (n-1)d$
Sum of Arithmetic Series	$S_n = \frac{n}{2}(a_1 + a_n)$
Finite Geometric Series	$\sum_{j=1}^{n} ar^{j-1} = \frac{a(1-r^n)}{1-r}$
Infinite Geometric Series	$\sum_{j=1}^{\infty} ar^{j-1} = \frac{a}{1-r}$
Binomial coefficients	$\binom{n}{r} = \frac{n!}{r!\cdot(n-r)!}$
Binomial Theorem	$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r}b^r$

1) Graph the following function. Clearly label all asymptotes and the x and y-intercepts.

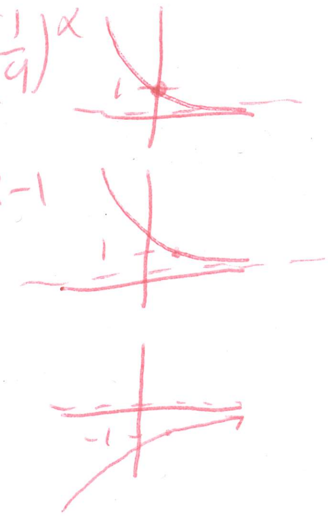
$$f(x) = -\left(\frac{1}{4}\right)^{x-1} + 1$$



Base: $y = \left(\frac{1}{4}\right)^{x}$

$\left(\frac{1}{4}\right)^{x-1}$

$-\left(\frac{1}{4}\right)^{x-1}$



$X_{int} \Rightarrow \quad 0 = -\left(\frac{1}{4}\right)^{x-1} + 1$

$\left(\frac{1}{4}\right)^{x-1} = 1$

$\log_{1/4}\left(\frac{1}{4}\right)^{x-1} = \log_{1/4}(1)$

$x - 1 = 0$

$x = 1$

$Y_{int} \Rightarrow x = 0$

$y = -\left(\frac{1}{4}\right)^{0-1} + 1$

$y = -\left(\frac{1}{4}\right)^{-1} + 1$

$y = -4 + 1 = 3$

2) (a) Combine the logarithms into one logarithm:

$$3\log_2(x) - 4\log_2(x+3) + \log_2(y) - \log_2(2)$$

(b) Solve following exponential equation for x:

$$4^{x+2} = \left(\frac{1}{32}\right)^{1-x}$$

(a) $\underbrace{\log_2(x^3) - \log_2((x+3)^4)} + \log_2(y) - \log_2(2)$

$= \underbrace{\log_2\left(\frac{x^3}{(x+3)^4}\right) + \log_2(y)} - \log_2(2)$

$= \log_2\left(\frac{x^3 y}{(x+3)^4}\right) - \log_2(2)$

$= \boxed{\log_2\left(\frac{x^3 y}{2(x+3)^4}\right)}$

(b) $4^{x+2} = \left(\frac{1}{32}\right)^{1-x}$

$(2^2)^{x+2} = (2^{-5})^{1-x} \Rightarrow 2^{2x+2} = 2^{-5+5x}$

$\Rightarrow 2x+4 = 5x-5$

$9 = 3x$

$\boxed{3 = x}$

3) Solve the following system of equations.

$$\begin{cases} x & -\ 6y & +\ 4z & = & -12 \\ x & +\ y & -\ 4z & = & 12 \\ 2x & +\ 2y & +\ 5z & = & -15 \end{cases}$$

Ans: $\boxed{\begin{array}{l} x = 0 \\ y = 0 \\ z = -3 \end{array}}$

Can solve via any method

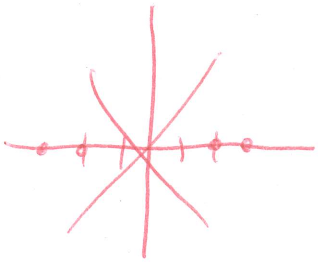
Answers vary

$$A = \left[\begin{array}{ccc|c} 1 & -6 & 4 & -12 \\ 1 & 1 & -4 & 12 \\ 2 & 2 & 5 & -15 \end{array}\right]$$

Augmented matrix

4) Write an equation for the ellipse having foci at (0,2) and (0,-2) , and vertices at (0, 3), and (0, -3).

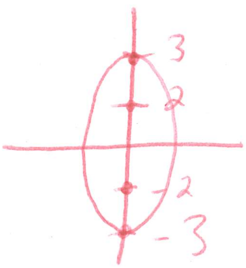
Center is midpoint of vertices or foci



$d_1 = 3-(-3) = 6 \Rightarrow r_1 = 3$

$d_2 = 2-(-2) = 4 \Rightarrow r_2 = 2$

$\Rightarrow b = 3, \; c = 2 \Rightarrow b^2 = 9, \; c^2 = 4$



$\Rightarrow (h,k) = (0,0)$

$c^2 = a^2 - b^2$

$4 = a^2 - 9$

$13 = a^2 \Rightarrow a = \pm\sqrt{13} \approx 3.6$

$$\Rightarrow \boxed{\frac{x^2}{13} + \frac{y^2}{9} = 1}$$

5) Write the following equation for the hyperbola in standard form. Then identify the center and the values of a and b.

$$9x^2 - 4y^2 - 36x + 48y - 144 = 0$$

$(9x^2 - 36x) + (-4y^2 + 48y) = 144$

$9(x^2 - 4x) - 4(y^2 - 12y) = 144$

$9(x^2 - 4x + 4) - 4(y^2 - 12y + 36) = \cancel{144} + 36 - \cancel{144}$

$9(x-2)^2 - 4(y-6)^2 = 36$

$$\boxed{\frac{(x-2)^2}{4} - \frac{(y-6)^2}{9} = 1}$$

$\boxed{a = 2 \quad b = 3 \quad (h,k) = (2,6)}$

6) (a) Find the general term in the sequence:

$$\{a_n\} = -\frac{3}{2}, \frac{7}{4}, -\frac{11}{8}, \frac{15}{16}, -\frac{19}{32}, \cdots$$

(b) Find the following sum using the general term found in part (a):

$$\sum_{n=1}^{4} \{a_n\}$$

(a) Sign change $-, +, -, +, -, \ldots \Rightarrow (-1)^n$ n starts at 1

Numerator:

n	1	2	3	4	5
?	3	7	11	15	19

$\Rightarrow 4n-1$

Denominator: 2^n $\Rightarrow a_n = \dfrac{(-1)^n(4n-1)}{2^n}$ $n \geq 1$

(b) $\displaystyle\sum_{n=1}^{4} a_n = -\frac{3}{2} + \frac{7}{4} - \frac{11}{8} + \frac{15}{16}$

$= -\dfrac{24}{16} + \dfrac{28}{16} - \dfrac{22}{16} + \dfrac{15}{16}$

$= \dfrac{-24+28-22+15}{16}$

$= \boxed{-\dfrac{3}{16}}$

7) Consider the sequence $\{a_n\} = \{3, -1, \frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, ...\}$.

a) Identify the type of sequence $\{a_n\}$.
b) What is the value of the term a_{11}? (Leave the answer as a fraction)
c) Find the sum of the following series: $\sum_{n=1}^{\infty} a_n$

(a) $\frac{a_2}{a_1} = \frac{-1}{3}$ $\quad \frac{a_3}{a_2} = \frac{1/3}{-1} = -\frac{1}{3}$ $\quad \frac{a_4}{a_3} = \frac{-1/9}{1/3} = -\frac{3}{9} = -\frac{1}{3}$

$\Rightarrow$ geometric

(b) $a_n = ar^{n-1}$

$a_{11} = 3\left(-\frac{1}{3}\right)^{11-1}$

$= 3\left(-\frac{1}{3}\right)^{10}$

$= \frac{3}{3^{10}} = \boxed{\frac{1}{3^9}}$

(c) $\sum_{n=1}^{\infty} 3\left(-\frac{1}{3}\right)^{n-1}$ $\quad \Rightarrow a = 3$, $r = -1/3$ $\Rightarrow |r| < 1$

$= \frac{a}{1-r} = \frac{3}{1-(-1/3)} = \frac{3}{\frac{4}{3}} = \cancel{3} \cdot \frac{3}{4} = \boxed{\frac{9}{4}}$

8) Use induction to prove:

$$S_n = \sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

Proof: $n=1 \Rightarrow S_1 = 1$ and $\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = 1$

for $n=1$ ✓

Assume true for $k \Rightarrow 1+2+3+\cdots+k = \frac{k(k+1)}{2}$ (☆)

Prove for $n=k+1$

$$(☆) \Rightarrow 1+2+3+\cdots+k+(k+1) = \frac{k(k+1)}{2} + k+1$$

$$= \frac{k^2+k+2k+2}{2}$$

$$= \frac{k^2+3k+2}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$ ✓

True for all $n \geq 1$

by induction

9) Use the binomial theorem to determine the 7^{th} term in the expansion of $(2x + y^4)^{10}$

$\Rightarrow n = 10,\ k = 7$

binomial term is $\binom{n}{k-1}$ for k^{th} term

$$\binom{10}{7-1} = \binom{10}{6} = \frac{10!}{6!\,4!} = \frac{(10\cdot 9\cdot 8\cdot 7)\cancel{6!}}{\cancel{6!}(4\cdot 3\cdot 2)} = \frac{\overset{5}{\cancel{10}}\cdot 9\cdot \overset{2}{\cancel{8}}\cdot 7}{\cancel{4}\cdot 3\cdot \cancel{2}}$$

$$= \frac{5\cdot \overset{3}{\cancel{9}}\cdot 2\cdot 7}{\cancel{3}} = 7\cdot 5\cdot 3\cdot 2 = 210$$

$$\cancel{\binom{n}{k-1}} \qquad \binom{n}{k-1} a^{n-(k-1)} b^{k-1} \qquad a = 2x,\ b = y^4$$

$$= \binom{10}{6}\cdot(2x)^{10-(7-1)}\cdot(y^4)^{7-1}$$

$$= \binom{10}{6}\cdot(2x)^4 (y^4)^6$$

$$= 210\cdot 2^4 x^4\cdot y^{24}$$

$$= \boxed{3360\, x^4\cdot y^{24}}$$

10) Suppose you flip a biased coin so that the probability of getting heads is $a = \frac{3}{4}$, or 75 %. What is the probability of getting heads 2 times and tails once? (You can leave your answer as a fraction).

Possible outcomes:

HHH	TTT
HHT	TTH
HTH	THT
THH	HTT

$(2)(2)(2) = 8 \Rightarrow$

3 ways to do this

prob of heads $= \frac{3}{4}$

prob of tails $= \frac{1}{4}$

$\Rightarrow 3 \cdot \left(\frac{3}{4}\right)^2 \cdot \left(\frac{1}{4}\right) = \boxed{\frac{27}{64}}$

$\approx \boxed{42\ \%}$

"Math" way there is $_3C_2$ or $\binom{3}{2}$ ways to get 2 heads, prob of getting heads $= \frac{3}{4} = p$

prob of getting tails $= \frac{1}{4} = q$

$$\Rightarrow \text{Prob}(2\text{ heads}) = \binom{3}{2} p^2 q^1$$

$$= 3\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)$$

$$= 3 \cdot \frac{9}{16} \cdot \frac{1}{4} = \boxed{\frac{27}{64}}$$

THIS PAGE IS LEFT BLANK FOR ANY SCRATCH WORK

END OF TEST