§1 - **Review**

Solving Equations

Example 0.1. $3y + 2[5(y-4) - 2] = 5y + 6(7+y) - 3$

Example 0.2. $(2y-3)^{\frac{1}{3}} - (4y+5)^{\frac{1}{3}} = 0$

Solving for a variable

Example 0.3. $a^2 + b^2 = c^2$ *Solve for a*

Example 0.4. $A = P + Prt$ *Solve for r*

Example 0.5. $A = \pi r^2$ *Solve for r*

Solving Inequalities

Example 0.6. $-2(x+3) < 10$ *Solve for x in interval notation*

Example 0.7. $4x - 1 \leq 3x - 4$ *Solve for x in interval notation*

Example 0.8. $-3 \leq 2x + 5 < 17$ *Solve for x in interval notation*

Example 0.9. $|x - 13| + 4 \leq 5$ *Solve for x in interval notation*

Word Problem

Example 0.10. *How much 80% antifreeze solution should be mixed with 2 gallons of 50% antifreeze solution to make a 60% antifreeze solution?*

§2.4 - **Linear Equations in Two Variables**

There are three forms for writing the equation of a line. They are as follows:
A linear equation can be written in standard form as: $ax + by = c$
A linear equation can be written in point-slope form as: $y - y_1 = m(x - x_1)$
A linear equation can be written in slope-intercept form as: $y = mx + b$

Example 0.11. *Graph the following lines:*

$$\begin{aligned} 2x + 3y &= 6 \\ x &= 3 \\ y &= 2 \end{aligned}$$

Example 0.12. *Find the slope of the three lines in the above problem.*

Example 0.13. *Find the slope of the line passing through the points $(-3, -2)$ and $(2, 5)$.*

Example 0.14. *Graph a line with slope -4 and passes through the point $(2, -3)$.*

The average rate of change of a function, $f(x)$ between two points, $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is given by the following equation:

$$\text{Average Rate of Change} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Example 0.15. *Find the average rate of change of the function* $f(x) = x^2 - 1$ *between the points* $x_1 = -2$ *and* $x_2 = 0$.

Example 0.16. *Solve the following equations and inequalities by graphing:*

$$2x - 3 = x - 1$$
$$2x - 3 < x - 1$$
$$2x - 3 > x - 1$$

Example 0.17. *Solve* $6x - 2(x + 2) - 5 \leq 0$ *by graphing.*

§2.5 - **Applications of Linear Equations**

Example 0.18. *Using the point slope formula, write the equation of the line passing through* $(4, -6)$ *and* $(1, -2)$.

If m_1 and m_2 are slopes of 2 non-vertical parallel lines, then $m_1 = m_2$.
If m_1 and m_2 are slopes of 2 non-vertical perpendicular lines, then $m_1 = -\frac{1}{m_2}$ or $m_1 m_2 = -1$.

Example 0.19. *For each of the following slopes* m_1, *write the slope* m_2 *that is perpendicular to it:*

$$m_1 = 2$$
$$m_1 = -3$$
$$m_1 = -\frac{1}{4}$$

Example 0.20. *Write an equation of a line passing through* $(-4, 1)$ *and parallel to the line* $x + 4y = 3$.

Example 0.21. *Write an equation of a line passing through* $(-3, 2)$ *and parallel to the line* $x + 3y = 6$.

Example 0.22. *Write an equation of a line passing through* $(2, -3)$ *and perpendicular to the line* $y = \frac{1}{2}x - 4$.

Example 0.23. *Write an equation of a line passing through* $(-8, -4)$ *and perpendicular to* $6y - x = 18$.

Linear Cost Function: $C(x) = mx + b$
Revenue Function: $R(x) = px$
Profit Function: $P(x) = R(x) - C(x)$

Where m is the variable cost, b is the fixed cost, x is the number of items, and p is the selling price.

Example 0.24. *A family phone plan has a monthly base price of* \$99 *plus* \$12.99 *for each additional phone added to the plan. Write the linear cost function* $C(x)$. *Compute* $C(4)$. *What does this answer mean?*

Example 0.25. *An art show vendor sells lemonade for* \$2.00 *per cup. The cost to rent the booth at the show costs* \$120. *Supplies to make and serve lemonade cost* \$.50 *per cup of lemonade.*
a) Write the cost function.
b) Write the revenue function.
c) Write the profit function.
d) How much profit is made if 50 cups are sold?
e) How much profit is made if 128 cups are sold?
f) How many cups must be sold for the vendor to break even?

Example 0.26. *If the fixed cost of a business is* \$2275, *its variable cost is* \$34.50, *and the price it sells each item is* \$80, *then write the three functions* $C(x), R(x)$, *and* $P(x)$.

Example 0.27. *Jorge borrows* \$2400 *from his grandmother and pays the money back at a rate of* \$150 *per month.*
a) Write the linear function $L(x)$ *for the amount of money that Jorge still owes his grandmother at* x *months.*
b) Calculate $L(12)$ *and explain its meaning.*

Example 0.28. *A car has a 15 gallon tank for gas and gets 30 miles to the gallon on a highway when driving 60 miles per hour. If he starts with a full tank of gas (15 gallons), and travels 450 miles at 60 miles per hour, then*
a) Write the $G(t)$ function for the amount left in the tank.
b) Find $G(4.5)$ and explain its meaning.

Example 0.29. *A dance studio has a fixed cost of* \$1500*. The studio charges* \$60 *for each lesson. The studio has a variable cost of* \$35 *to pay instructors.*
a) Write the cost function.
b) Write the revenue function.
c) Write the profit function.
d) Determine the number of lessons to make a profit.
e) If 82 lessons are given in one month how much does the studio make?

§2.6 - **Transformations**

Below is a chart that summarizes all of the transformations we covered in class. The general formula is in the second column, and there is an example given for each case for the function $f(x) = |x|$ (the absolute value function). You can use the second column as a guide to work with any function that you are given. In class we will work with $f(x) = |x|$, $f(x) = x^2$, and $f(x) = x^3$.

Transformations of the Absolute Value Parent Function f(x)=|x|

Transformation	F (x) Notation	Examples	
Vertical translation	f(x) + k	Y = |x| + 3 Y = |x| - 4	3 units up 4 units down
Horizontal translation	f(x – h)	Y = |x-2| Y = |x+1|	2 units right 1 unit left
Vertical stretch/ compression	af(x)	Y = 6|x| Y = ½ |x|	vertical stretch by 6 vertical compression by 1/2
Horizontal stretch/compression	F(1/bx)	Y = |1/5x| Y= |3x|	horizontal stretch by 5 horizontal compression by 1/3
Reflection	-f(x) f(-x)	Y = -|x| Y = |-x|	across x-axis across y-axis

FIGURE 1. Transformation Chart - From http://hellermaayanotmath.wikispaces.com

Below are a few examples of using transformations that may be helpful for studying.

Example 0.30. *Apply a horizontal translation 3 units left and a vertical translation of 4 units down for the graph of $f(x) = x^2$, and write the formula for this graph.*

Example 0.31. *Apply a horizontal translation 2 units right, a vertical translation of 5 units up, and a reflection over the x-axis for the graph of $f(x) = |x|$, and write the formula for this graph.*

Example 0.32. *Sketch the graph of $f(x) = (x + 5)^5 - 2$ using graph transformations.*

Example 0.33. *Sketch the graph of $f(x) = (x + 1)^3 + 2$ using graph transformations.*

Example 0.34. *Sketch the graph of $f(x) = -|x - 3| + 4$ using graph transformations.*

Example 0.35. *Sketch the graph of $f(x) = -2(x + 1)^2 - 5$ using graph transformations.*

Example 0.36. *Sketch the graph of $f(x) = \sqrt{x + 5}$ using graph transformations.*

§2.7 - **Graphs of Functions and Piecewise Functions**

Symmetry Test:

1) The graph of a function is symmetric about the y-axis if we substitute $-x$ for x, and reduce, then we get the same function we started with.
2) The graph of a function is symmetric about the x-axis if we substitute $-y$ for y, and reduce, then we get the same function we started with.
3) The graph of a function is symmetric about the origin if we substitute $-x$ for x AND $-y$ for y, and reduce, then we get the same function we started with.

Check the above 3 conditions on the following functions to check for symmetry:

Example 0.37. $y = |x|$

Example 0.38. $x = y^2 - 4$

Example 0.39. $y = x^2$

Example 0.40. $y = x^3$

Even and Odd Test:

1) The graph of a function is even if $f(-x) = f(x)$ for all x in the domain of the function. The graph of an even function is symmetric about the y-axis.
2) The graph of a function is odd if $f(-x) = -f(x)$ for all x in the domain of the function. The graph of an odd function is symmetric about the origin
3) If the function $f(x)$ does not satisfy either condition above, the function is neither even nor odd.

Check the above 3 conditions on the following functions to check if the function is even, odd, or neither:

Example 0.41. $f(x) = -2x^4 + 5|x|$

Example 0.42. $f(x) = 4x^3 - x$

Example 0.43. $f(x) = -x^5 + x^3$

Example 0.44. $f(x) = x^2 - |x| + 1$

Example 0.45. $f(x) = 2|x| + x$

Example 0.46. $f(x) = x^4 + x^2 + x + 1$

Piecewise Functions

Evaluate the following function at the given points:

$$f(x) = \begin{cases} -x-1 & \text{for} \quad -4 \leq x < -1 \\ -3 & \text{for} \quad -1 \leq x < 2 \\ \sqrt{x-2} & \text{for} \quad x \geq 2 \end{cases}$$

a) $f(-3)$
b) $f(-1)$
c) $f(2)$
d) $f(6)$

Graphing Piecewise Functions

Here are some guided steps that you can use to always graph piecewise functions accurately:

1) Draw your x and y axes (if none are provided for you).
2) Plot your points that are at end of the intervals first (the ones that are given next to each piece (ex. $-4, -1, 2$ in the previous example). Be sure to pay attention whether the dot must be closed or open.
3) Draw lightly a vertical dotted line through each point. This will let you know that each segment of the graph can only be located in the regions that you have sliced the plane into.
4) Draw the graph the question indicates in each region.
5) The graph should be completed in each section, this is your piecewise graph.

Here are some examples to try. Graph the following piecewise functions:

Example 0.47.

$$f(x) = \begin{cases} x+3 & \textit{for} \quad x < -1 \\ x^2 & \textit{for} \quad -1 \leq x < 2 \end{cases}$$

Example 0.48.

$$f(x) = \begin{cases} |x| & \textit{for} \quad -4 \leq x < 2 \\ x^2 & \textit{for} \quad x \geq 2 \end{cases}$$

Example 0.49.

$$f(x) = \begin{cases} -x+1 & \textit{for} \quad x < 1 \\ \sqrt{x} & \textit{for} \quad 1 \leq x < 4 \end{cases}$$

Example 0.50.

$$f(x) = \begin{cases} x^2+1 & \textit{for} \quad x < -2 \\ 2x+3 & \textit{for} \quad x \geq 2 \end{cases}$$

Example 0.51.

$$f(x) = \begin{cases} 2 & \textit{for} \quad x \leq -1 \\ -2x & \textit{for} \quad x > -1 \end{cases}$$

Example 0.52.

$$f(x) = \begin{cases} 3x+3 & \textit{for} \quad x < 1 \\ x^2 & \textit{for} \quad 1 \leq x < 2 \\ -x-1 & \textit{for} \quad x \geq 2 \end{cases}$$

Greatest Integer Function

The greatest integer function (or floor function) is a special piecewise defined graph. it is defined as

$$f(x) = [[x]] \quad \text{where } [[x]] \text{ denotes the greatest integer less than or equal to } x$$

Evaluate the greatest integer function at the given points:

a) $f(1)$
b) $f(1.7)$
c) $f(-2.2)$
d) $f(3.5)$

Increasing, Decreasing, and Constant Functions

Suppose that I is an interval contained within the domain of $f(x)$

1) $f(x)$ is an increasing on I if $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in I.
2) $f(x)$ is an decreasing on I if $f(x_1) > f(x_2)$ for all $x_1 < x_2$ in I.
3) $f(x)$ is an increasing on I if $f(x_1) = f(x_2)$ for all x_1 and x_2 in I.

Find where the following functions are increasing, decreasing, or constant:

Example 0.53. $f(x) = x^2$

Example 0.54. $f(x) = |x+1|$

Example 0.55. $f(x) = -2x+1$

Relative Minimum and Maximum Values

Let $f(x)$ be a function, and $x = a$ and $x = b$ be two points. Then $f(a)$ is a relative maximum of $f(x)$ if there exists an open interval containing a such that $f(a) \geq f(x)$ for all x in the interval. Alternatively, $f(b)$ is a relative minimum of $f(x)$ if there exists an open interval containing b such that $f(b) \leq f(x)$ for all x in the interval. An open interval is an interval that does not include the endpoints.

Find where the following functions have relative maxima and minima:

Example 0.56. $f(x) = x^2$

Example 0.57. $f(x) = |x+1|$

Example 0.58. $f(x) = -(x-2)^2 + 1$

§3.1 - **Quadratic Functions**

Definition 0.1. *A* ***quadratic function*** *is of the form*

$$f(x) = ax^2 + bx + c$$

Definition 0.2. *The maximum or minimum of the parabola is called a* ***vertex****.*

Definition 0.3. *The vertical line that passes through the vertex is called the* ***axis of symmetry****.*

The following proof is to show how to get the vertex form of the parabola in general. We will have to complete the square.

$$\begin{aligned}
f(x) &= ax^2 + bx + c \\
&= a\left(x^2 + \frac{b}{a}x\right) + c \\
&= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - a\left(\frac{b^2}{4a^2}\right) \\
&= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \\
&= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \\
&= a(x-h)^2 + k
\end{aligned}$$

where we define $h = -\frac{b}{2a}$ and $k = \frac{4ac-b^2}{4a}$. So then we have that

$$\text{Vertex} = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

$$\text{Axis of Symmetry} \Rightarrow x = -\frac{b}{2a}$$

Definition 0.4. *If we have a quadratic function $f(x) = ax^2 + bx + c$, then*
(i) $a > 0$ implies that the vertex is a minimum.
(ii) $a < 0$ implies that the vertex is a maximum.

Definition 0.5. *If we have a quadratic function $f(x) = ax^2 + bx + c$, then*
(i) If $b^2 - 4ac > 0 \Rightarrow$, then f has 2 real roots.
(ii) If $b^2 - 4ac = 0 \Rightarrow$, then f has 1 real root.
(iii) If $b^2 - 4ac < 0 \Rightarrow$, then f has no real roots (does not cross the x-axis).

Below are a few examples of writing the quadratic in vertex form.

Example 0.59. *Rewrite the function $f(x) = x^2 - 8x + 7$ in vertex form.*

Example 0.60. *Rewrite the function $f(x) = x^2 + 6x + 1$ in vertex form.*

Example 0.61. *Rewrite the function $f(x) = 3x^2 + 12x + 2$ in vertex form.*

Example 0.62. *Rewrite the function $f(x) = 4x^2 - 40x + 13$ in vertex form.*

§3.2 - **Introduction to Polynomials**

Definition 0.6. *A* ***polynomial function*** *is of the form*

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$$

Definition 0.7. *The* ***degree*** *of a polynomial is the highest value exponent of the function $f(x)$ above.*

Here is a good chart to summarize the end behavior of polynomials that we covered in class. This includes all the necessary information (this is also called the leading term test in the textbook) to know along with some examples of each.

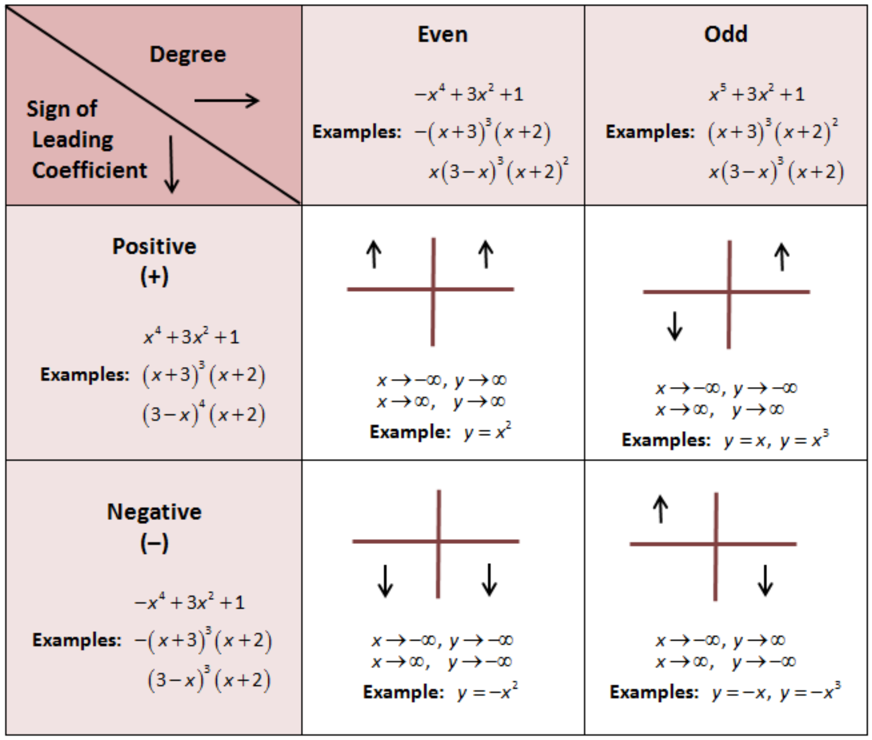


FIGURE 2. End Behavior Chart - From http://www.shelovesmath.com/algebra/advancedalgebra/graphing-polynomials/

Definition 0.8. *The* ***multiplicity*** *of $f(x) = (x-a)^n$ is the power n. If n is odd, then the graph* ***crosses*** *at the x-intercept $(a, 0)$. If n is even, then the graph* ***touches*** *at the x-intercept $(a, 0)$.*

Theorem 0.1. *Intermediate Value Theorem: Let f be a polynomial function. For $a < b$, if $f(a)$ and $f(b)$ have opposite signs, then f has at least one zero on the interval $[a, b]$.*

Example 0.63. *Sketch the graph of $f(x) = x(x+1)^2(x-1)^2$.*

Example 0.64. *Sketch the graph of $f(x) = (x+5)^4(x+1)^3(x-1)^2$.*

Example 0.65. *Sketch the graph of $f(x) = -(x+1)^2(x-4)^2$.*

Example 0.66. *Sketch the graph of $f(x) = -x(x+3)^4(x-3)^2$.*

Example 0.67. *Sketch the graph of $f(x) = x^4 - 2x^2$.*

Example 0.68. *Sketch the graph of $f(x) = x^3 - 9x$.*

Example 0.69. *Sketch the graph of $f(x) = 9x^5 + 9x^4 - 25x^3 - 25x^2$.*

§3.3 - **Polynomial Division and Factoring Theorems**

Definition 0.9. *We can write a polynomial $f(x)$ as the following if $d(x) \neq 0$ and the degree of $d(x)$ is less than or equal to the degree of $f(x)$:*

$$f(x) = d(x)q(x) + r(x)$$

Where $f(x)$ is the dividend, $d(x)$ is the divisor, $q(x)$ is the quotient, and $r(x)$ is the remainder.

Example 0.70. *Use long division to divide:* $(6x^3 - 5x^2 - 3) \div (3x + 2)$.

Example 0.71. *Use long division to divide:* $(3x^4 + 2x^3 + 4x^2 + x - 5) \div (3x + 2)$.

Example 0.72. *Use long division to divide:* $(4x^3 - 23x + 3) \div (2x - 5)$.

Example 0.73. *Use long division to divide:* $(2x^4 - 3x^3 + 5x^2 - 7x + 1) \div (x^2 + 3)$.

Example 0.74. *Use long division to divide:* $(2x^2 + 3x - 14) \div (x - 2)$.

Example 0.75. *Use long division to divide:* $(3x^2 - 14x + 15) \div (x - 3)$.

Example 0.76. *Use synthetic division to divide:* $(3x^2 - 14x + 15) \div (x - 3)$.

Example 0.77. *Use synthetic division to divide:* $(-10x^2 + 2x^3 - 5) \div (x - 4)$.

Example 0.78. *Use synthetic division to divide:* $(4x^3 - 28x - 7) \div (x - 3)$.

Example 0.79. *Use synthetic division to divide:* $(x^4 + 4x^3 - 2x + 18) \div (x + 3)$.

Example 0.80. *Use synthetic division to divide:* $(2x^4 + 7x^3 - 3x + 5) \div (x + 1)$.

Example 0.81. *Given $f(x) = x^4 + 6x^3 - 12x^2 - 30x + 35$, use synthetic division and the remainder to find $f(2)$ and $f(-7)$.*

Example 0.82. *Given $f(x) = x^4 + x^3 - 6x^2 - 5x - 15$, use synthetic division and the remainder to find $f(5)$ and $f(-3)$.*

Example 0.83. *Given $f(x) = 2x^4 - 4x^2 - 13x - 9$, use synthetic division and the remainder to determine if $c = 4$ is a zero of $f(x)$.*

Example 0.84. *Given $f(x) = x^3 + x^2 - 3x - 3$, use synthetic division and the remainder to determine if $c = \sqrt{3}$ is a zero of $f(x)$.*

Example 0.85. *Given $f(x) = x^3 + x + 10$, use synthetic division and the remainder to determine if $c = 1 + 2i$ is a zero of $f(x)$.*

Example 0.86. *Given $f(x) = x^4 - x^3 - 11x^2 + 11x + 12$, use synthetic division and the factor theorem to determine if $(x - 3)$ and $(x + 2)$ are factors $f(x)$.*

Example 0.87. *Given $f(x) = 2x^4 - 13x^3 + 10x^2 - 25x + 6$, use synthetic division and the factor theorem to determine if $(x - 6)$ and $(x + 3)$ are factors $f(x)$.*

Example 0.88. *Factor $f(x) = 3x^3 + 25x^2 + 42x - 40$, given that -5 is a zero of $f(x)$. Then solve the equation $3x^3 + 25x^2 + 42x - 40 = 0$.*

Example 0.89. *Factor $f(x) = 2x^3 + 7x^2 - 14x - 40$, given that -4 is a zero of $f(x)$. Then solve the equation $2x^3 + 7x^2 - 14x - 40 = 0$.*

Example 0.90. *Write a polynomial $f(x)$ of degree three that has roots $x = 1$, $x = 2$, and $x = 3$.*

Example 0.91. *Write a polynomial $f(x)$ of degree three that has roots $x = \frac{1}{3}$, $x = \sqrt{6}$, and $x = -\sqrt{6}$.*

Example 0.92. *Write a polynomial $f(x)$ of degree three that has roots $x = 2$, $x = 1 + i$, and $x = 1 - i$.*

Section 3.4 - Zeros of Polynomials

Rational Root Theorem: If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ has integer coefficients and $a_n \neq 0$ if $\frac{p}{q}$ is a rational zero of f, then p is a factor of a_0
q is a factor of a_n $\quad \frac{p}{q} = \pm\frac{\dots}{\dots}$

Ex) $f(x) = -2x^5 + 3x^2 - 2x^2 + 10$] Test

~~$f(x) = -4x^4 + 5x^3 - 7x^2 + 8$~~ $\rightarrow$

$f(x) = x^3 - 4x^2 + 3x + 2 \rightarrow 2,\ 1 \pm \sqrt{2}$

$f(x) = x^3 - x^2 - 4x - 2 \longrightarrow -1,\ 1 \pm \sqrt{3}$

Ex) $f(x) = 2x^4 + 5x^3 - 2x^2 - 11x - 6 \rightarrow x = -2, -1, \frac{3}{2}$

$f(x) = 2x^4 + 3x^3 - 15x^2 - 32x - 12 \rightarrow x = -2, -\frac{1}{2}, 3$

Ex) $\left.\begin{array}{l} f(x) = x^4 - 2x^2 - 3 \\ f(x) = x^4 - x^2 - 20 \end{array}\right\}$ non-exmapls $\quad (x^2-3)(x^2+1)$

$(x^2-5)(x^2+4)$

Let $x^2 = z$ $\quad z^2 - 2z - 3$

$z^2 - z - 20$

Fund Thm of Algebra: $f(x)$ has degree $n \geq 1$ with complex coeffs, then f has at least 1 complex root

\# of zeros: $f(x)$, $n \geq 1$, complex coeffs, f has exactly n complex roots. (including multiplicity)

$f(x) = x^4 - 6x^3 + 28x^2 - 18x + 75$ and $3-4i$

$3+4i, \pm i\sqrt{3}$

Descartes' Rule of Signs

$f(x) = x^5 - 6x^4 + 12x^3 - 12x^2 + 11x - 6$

Let $f(x)$ be a polynomial with real coeffs. and non-zero constant term

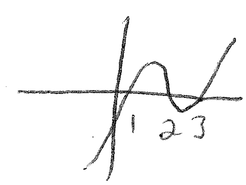
1) # of pos real zeros is either
   - ~~#~~ same as # of sign changes in $f(x)$
   - less than # of sign changes in $f(x)$ by + integer

2) # of neg real zeros is either
   - # of sign change in ~~f(x)~~ $f(-x)$
   - less than the # of sign change in $f(-x)$ by pos. integer

5 sign change $\Rightarrow$ + real zeros : 5, 3, 1

$f(-x) = -x^5 - 6x^4 - 12x^3 - 12x^2 - 11x - 6$

0 sign changes $\Rightarrow$ no neg real roots



Find zeros of $f(x) = 2x^5 + x^4 + 9x^2 - 32x + 20$

1, 1, $-\frac{5}{2}$ sign change

$f(x) = x^5 + 6x^3 - 2x^2 - 27x - 18$

Section 3.5 – Rational Functions

Let $p(x)$ and $q(x) \neq 0$ be two functions.

$f(x) = \frac{p(x)}{q(x)}$ is called a rational function

Exs)

Function	Factored	Domain
$f(x) = \frac{1}{x}$	$f(x) = \frac{1}{x}$	$(-\infty, 0) \cup (0, \infty)$
$g(x) = \frac{5x^2}{2x^2+5x-12}$	$g(x) = \frac{5x^2}{(2x-3)(x+4)}$	$(-\infty, -4) \cup (-4, \frac{3}{2}) \cup (\frac{3}{2}, \infty)$
$k(x) = \frac{x+3}{x^2+4}$	$k(x) = \frac{x+3}{x^2+4}$	$\mathbb{R}$

Ex) $\frac{1}{x^2}$

$x \to c^+$, $x \to c^-$, $x \to \infty$, $x \to -\infty$

Vertical Asymptote: $f(x) = \frac{p(x)}{q(x)}$ no common factors

c is a real zero

$x = c$ is a vertical asymptote

Ex) $f(x) = \frac{2}{x-3}$, $g(x) = \frac{x-4}{3x^2+5x-2}$ $k(x) = \frac{4x^2}{x^2+4}$

Holes: $f(x) = \frac{2x^2+5x+3}{x+1} = \frac{(2x+3)(x+1)}{x+1}$

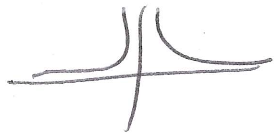
Horiz. Asymp: Picture: If degree of top < bottom

$y = 0$ is a horizontal asymp.

If degree of top = bottom, just

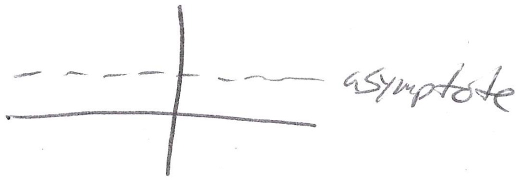
$y = \frac{\text{top coeff}}{\text{bottom coeff}}$

Ex) $f(x) = \dfrac{x^2 + \cdots}{x^4 + \cdots}$



bottom > top

$f(x) = \dfrac{4x^2 + \cdots}{3x^2 + \cdots}$

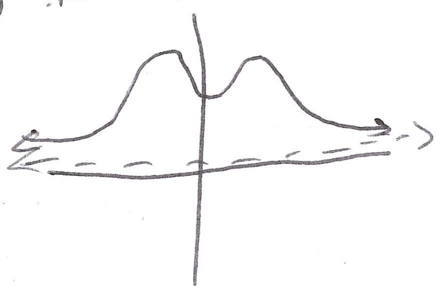


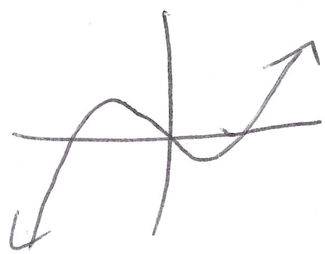
bottom = top

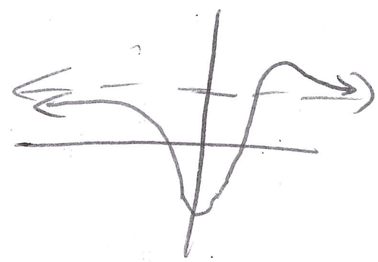
bottom < top
↳ blow up

Ex) $f(x) = \dfrac{8x^2+1}{x^4+1}$, $g(x) = \dfrac{2x^3-6x}{x^2+4}$, $h(x) = \dfrac{8x^2+9x-5}{2x^2+1}$

<u>Q:</u> ID the asymptotes



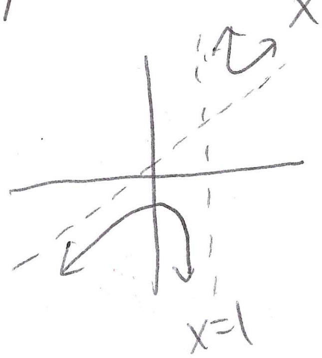


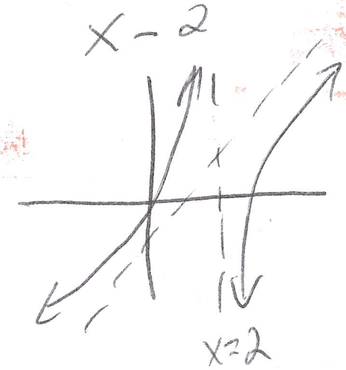


<u>Slant Asymptote:</u> • Rational function has slant asymp. if degree of numerator is <u>exactly one</u> greater than denom.

• To find, polynomial divide.

Ex) $f(x) = \dfrac{x^2+1}{x-1}$ $\quad f(x) = \dfrac{2x^2-5x-3}{x-2}$





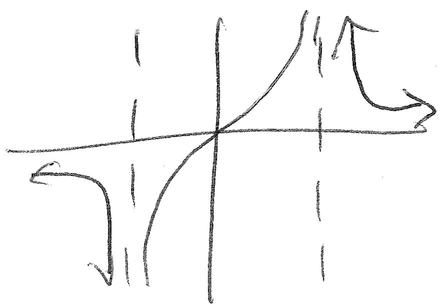
Graph Transformations on Rational Functions

$$f(x) = \frac{1}{(x+2)^2} + 3$$

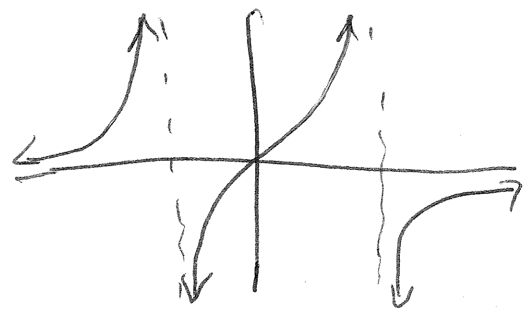
Steps

① Determine y int
② " x ints
③ Vert. asymptote
④ Horiz. asymptote
⑤ Slant asymptote
⑥ Symmetry
⑦ Sign Chart
⑧ Sketch

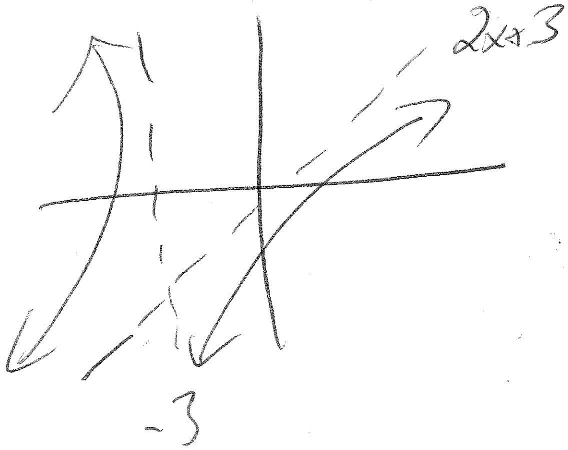
Ex) $f(x) = \dfrac{4x}{x^2-4}$



Ex) $g(x) = \dfrac{2x^2-3x-5}{x^2+1}$



Ex) $h(x) = \dfrac{2x^2+9x+4}{x+3}$



$x = -4, -\frac{1}{2}$

$= (2x+1)(x+4)$

3.6 - Quadratic / Rational Inequalities

Use sign chart:

0. Put all terms on one side; one side zero (combine to one fraction)
1. Find zeros
2. Find undef pts
3. Test in each region + or −
4. Check inequality
5. Interval Notation

Ex)
(1) $x^2-4x-12 \le 0$
(4) $x^2+x > 12$
(2) $2x^2 < x+10$
(5) $-x^2-x+6 < 0$
(3) $3x(x-1) > 10-2x$
(6) $2x(x-1) < 21-x$
(7) $x^4-12x \ge 8x^2-x^3$ $(-2, 0, 3)$

Rational Inequalities: ~~f(x) Constant~~

Ex) $\dfrac{4x-5}{x-2} \le 3$ Ex) $\dfrac{5-x}{x-1} \ge -2$

Ex) $\dfrac{x^2}{x^2+4} \ge 0$ Ex) $\dfrac{x^2}{x^2+4} < 0$

Ex) $\dfrac{5}{x-3} > \dfrac{3}{x+1}$ Ex) $\dfrac{(x+3)(x-5)}{3(x-1)} > 0$

Ex) $\dfrac{x^2+4x-45}{x+1} \le 0$

Section 2.8 - Functions and Function Composition

Sum: $(f+g)(x) = f(x)+g(x)$

Difference: $(f-g)(x) = f(x)-g(x)$

Product: $(f\cdot g)(x) = f(x)g(x)$

Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad g(x) \neq 0$

Ex) $f(x) = \sqrt{25-x^2} \quad g(x) = 5x$ Find the above.

Ex) $m(x) = 4x \quad n(x) = |x-1| \quad p(x) = \frac{1}{x+1}$

Find $(m-n)(2)$, $(m\cdot p)(1)$, $\left(\frac{p}{n}\right)(3)$

Ex) $g(x) = 2x$, $K(x) = \sqrt{x-1}$, $h(x) = x^2-4x$

Find $(g-h)(x)$, $(g\cdot K)(x)$, $\left(\frac{K}{h}\right)(x)$ and domains of each

Difference Quotient: $\frac{f(x+h)-f(x)}{h}$

Find Diff. Quotient for
$f(x) = 3x-5$
$f(x) = -2x^2+4x-1$
$f(x) = \sqrt{x}$
$f(x) = 1/x$

The composition of f and g is $f\circ g$ or $(f\circ g)(x) = f(g(x))$

Ex) $f(x) = x^2+2x \quad g(x) = x-4$

a) $f(g(6))$
b) $(f\circ g)(0)$
c) $g(f(-3))$
d) $(g\circ f)(5)$

Ex) $f(x) = 2x-6$ $\quad (f \circ g)(x)$ domains.

$g(x) = \frac{1}{x+4}$ $\quad (g \circ f)(x)$

Ex) $f(x) = \frac{1}{x-5}$

$g(x) = \sqrt{x-2}$ $\quad (f \circ g)(x)$ domains.

$(g \circ f)(x)$

Word Problems

Ex 9, p. 301

Decomposing functions

Let $h(x) = |2x-5|$ find f, g s.t $(f \circ g)(x) = h(x)$

Let $h(x) = \sqrt[3]{5x+1}$ find " " " "

Section 4.1 - Inverse Functions

Defn: A function is 1-1 if for a and b in the domain of f, if $a \neq b$, then $f(a) \neq f(b)$

So $f(a) = f(b)$ iff $a = b$

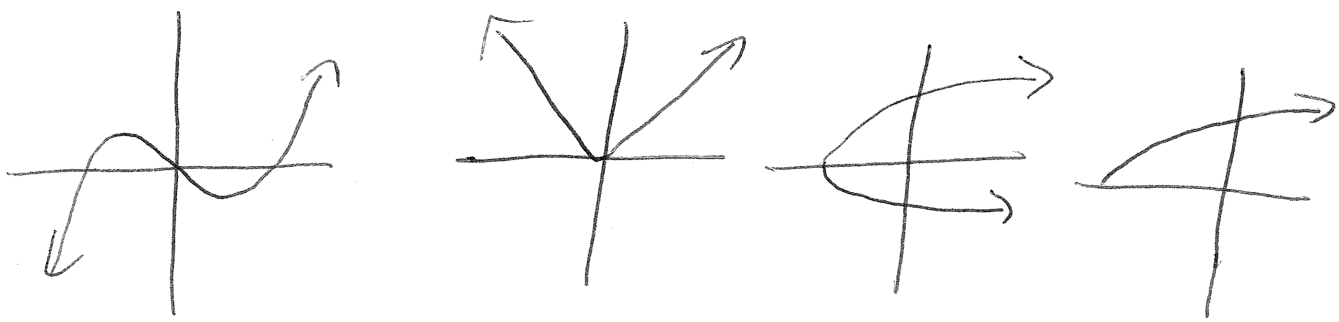
Ex) Is f 1-1?

$f = \{(1,4), (2,3), (-2,4)\}$

$f = \{(3,5), (4,5), (2,1)\}$

$f(x) = x^2$

Ex) Horizontal Line Test



Ex) Show if $f(x)$ is 1-1. $f(x) = 2x-3$

$f(x) = x^2+1$

Defn: Let f be 1-1. Then g is an inverse of f if

1. $(f \circ g)(x) = x$ for all x in domain of g
2. $(g \circ f)(x) = x$ " x in domain of f

Or $(f \circ f^{-1})(x) = x$

$(f^{-1} \circ f)(x) = x$

Note: $f^{-1} \neq \frac{1}{f}$

Ex) Let $f(x) = 100 + 12x$. Is $g(x) = \frac{x-100}{12}$ an inverse?

To find inverse,

1. Replace $f(x)$ by y
2. Switch y and x
3. Solve for y
4. Change y to $f^{-1}(x)$.

Ex) $f(x) = 3x - 1$ $\qquad f(x) = \frac{x-2}{x+2}$

$f(x) = 4x + 3$

$f(x) = \frac{3-x}{x+3}$ $\qquad f(x) = x^2 + 4 \quad x \geq 0$ (symmetric about $y = x$)

$f(x) = \sqrt{x-1}$ $\qquad f(x) = \sqrt{x+2}$

$f(x) = \frac{x+4}{2x-5}$ $\qquad$ Check

Section 4.2 - Exponential Functions

Defn: Let b be a real number, $b>0$ $b\neq 1$, then
$f(x) = b^x$ is an exponential function

Examples: $4^x, e^x, \left(\frac{1}{3}\right)^x, (\sqrt{2})^x$ **Non Ex:** $1^x, (-4)^x, x^2$

Graphs of $f(x) = b^x$

1. For $b>1$, exponential growth
2. For $0<b<1$, " decay
3. Domain: $(-\infty, \infty)$
4. Range: $(0, \infty)$
5. $y=0$ is an HA
6. $f(0) = b^0 = 1$ y int is $(0,1)$

Examples: $2^x, 5^x, \left(\frac{1}{2}\right)^x, e^x$ $e \approx 2.718$

Graph Transformations: $f(x) = ab^{x-h} + k$

$h>0$ right $k>0$ up $a<0$ reflect over x-axis
$h<0$ left $k<0$ down $0<|a|<1$ vert. shrink
$|a|>1$ vert stretch

Graph: $f(x) = 3^{x-2} + 4$ $f(x) = e^x$
$f(x) = 2^{x+2} - 1$
$f(x) = -e^{x-2} + 2$

Interest: $A = P + I = P + Prt$

from this we derive some equations
see p. 449-50 for details

Suppose P dollars invested.
at rate of r $/yr for t years

① $I = Prt$ Amount of simple interest I

② $A = P\left(1+\frac{r}{n}\right)^{nt}$ Amount after t years
n periods per year

③ $A = Pe^{rt}$ Amount after t years continuously

Ex 4 p 451

Ex) A sample originally having 1g of Ra 226, the amount

$A(t) = \left(\frac{1}{2}\right)^{t/1620}$ after t time

How much present after 1620 yr? 3240 yr? 4860 yr

^1 ^2 ^3

Ans: .5, .25, .125

Section 4.3 - Logarithmic Equations

Defn: Let b be a real number, $b \neq 1$ $b > 0$

$y = \log_b(x)$ is a logarithm function with base b

Ex: $\log_{10}(x)$, $\log(x)$, $\ln(x)$, $\log_2(x)$

$f(x) = b^x$
$y = b^x$
$x = b^y$
$\log_b(x) = y$

Note: $y = \log_b(x) \sim b^y = x$

Ex) $\log_2 16 = 4 \Leftrightarrow 16 = 2^4$

$\log_{10}\left(\frac{1}{100}\right) = -2 \Leftrightarrow \frac{1}{100} = 2^{-2}$

$\log_7 1 = 0 \Leftrightarrow 7^0 = 1$

Ex) Evaluate: $\log_4 16$, $\log_2 8$, $\log_{1/2} 8$

Note: $\log_{10}(x) = \log(x)$ $\log_e(x) = \ln(x)$

Ex) $\log(100)$, $\log\left(\frac{1}{10}\right)$, $\ln e^4$, $\ln\left(\frac{1}{e}\right)$

Rules: $\log_b 1 = 0$, $\log_b b = 1$, $\log_b b^x = x$, $b^{\log_b x} = x$

Graps of $f(x) = \log_b(x)$

1. $b > 1$ increasing
2. $b < 1$ decreasing
3. Domain: $(0, \infty)$
4. Range: $(-\infty, \infty)$
5. $x = 0$ is a VA
6. $f(1) = 0 \Rightarrow x = 1$ is xint

Ex's: $\log_{10}(x)$ $\log_{1/2}(x)$
$\ln(x)$

Graph Transformation: $f(x) = a\log_b(x-h) + K$

$h > 0$ ~~h~~ right

$h < 0$ left

$K > 0$ up

$K < 0$ down

$a < 0$ reflection over x

$0 < |a| < 1$ vert. shrink

$|a| > 1$ vert stretch

Graph:

$f(x) = \log(x-2) + 4$

$f(x) = \log_2(x+3) - 2$

$f(x) = -\ln(x-1) + 1$

Find domain of each

Section 4.4 - Logarithm Properties

Rules:

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$

$$\log_b(x^p) = p\log_b(x)$$

$\log(x+y) \neq$ add or sum of mult.

Ex) $\log_2(8x)$

$\ln(5xy)$

$\log_4(16a)$

Ex) $\log_3\left(\frac{c}{d}\right)$ $\quad \log_6\left(\frac{8}{t}\right)$

$\log\left(\frac{x}{1000}\right)$ $\quad \ln\left(\frac{e}{12}\right)$

Ex) $\ln\sqrt[5]{x^2}$ $\quad \log_5\sqrt[5]{x^4}$ $\quad \ln x^4$

Shortcut Rule: $\log\left(\frac{ab}{c}\right) = \log(a) + \log(b) - \log(c)$

Expand logs

Ex) $\log_2\left(\frac{z^3}{xy^5}\right)$ $\quad \log\sqrt[3]{\frac{(x+y)^2}{10}}$

$\ln\left(\frac{a^4 b}{c^9}\right)$ $\quad \log_5\left(\sqrt[3]{\frac{25}{(a+b)^2}}\right)$

Reverse: Combine to one log

Ex) $\log_2 560 - \log_2 7 - \log_2 5$ $\quad 3\log x - \frac{1}{3}\log y - \frac{2}{3}\log z$

$3\log a - \frac{1}{2}\log b - \frac{1}{2}\log c$ $\quad \frac{1}{3}\ln t + \ln(t^2-9) - \ln(t-3)$

$\frac{1}{2}\ln x + \ln(x^2 \pm 1) - \ln(x+1)$

Change of base formula

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Evaluate $\log_3(6) = \dfrac{\ln(6)}{\ln(3)} \approx 1.631$

$$\log_3(6) = \frac{\log(6)}{\log(3)} \approx 1.631$$

Section 4.5 - Exponential and Log Eqns

Solve:

$3^{2x-6} = 81$ $\qquad$ $27^{2w+5} = \left(\frac{1}{3}\right)^{2-5w}$

$25^{4-t} = \left(\frac{1}{5}\right)^{3t+1}$

$4^{2x-3} = 64$

Solve:

$7^x = 60$ $\qquad$ (Multiple solutions)

$5^x = 83$

Solve:

$4^{2x-7} = 5^{3x+1}$

$3^{5x-6} = 2^{4x+1}$

Solve:

$e^{2x} + 5e^x - 36 = 0$

$e^{2x} - 5e^x - 14 = 0$

Solve:

$\log_2(3x-4) = \log_2(x+2)$ $\qquad$ Always Check Answers

$\log_2(7x-4) = \log_2(2x+1)$

$4\log_3(2t-7) = 8$

$8\log_4(w+6) = 24$

$\log_2 x = 3 - \log_2(x-2)$

$2 - \log_7 x = \log_7(x-48)$

Ex) $\ln(x-4) = \ln(x+6) - \ln(x)$

$\ln x + \ln(x-8) = \ln(x-20)$

Ex) $A = P\left(1+\frac{r}{n}\right)^{nt}$

How long will it take \$4000 investment to double with a 4.5% intrest rate compounded monthly?

$t = 15.4$

Section 4.6 - Modeling with Exponential + Log Functions

Ex) Given $P = 100e^{kx} - 100$ solve for x (geology)
" $L = 8.8 + 5.1 \log D$ solve for D (astro)
$T = 78 + 272e^{-kt}$ find k
$S = 90 - 20\ln(t+1)$ find t

Let y be a variable changing exponentially
y_0 be initial condition for y at $t = 0$

1) If $k > 0$ $y = y_0 e^{kt}$ is exponential growth

$y = 2000e^{.06t}$

continuous compound interest.

2) If $k < 0$, $y = y_0 e^{kt}$ is exponential decay

$y = 100e^{-.165t}$

Radioactive treatment Iodine

Ex) $A = Pe^{rt}$ $P = 15,000$
$t = 3 \Rightarrow A = 19,356.92$
find rate : $r = .085$

Ex) $P(t) = P_0 e^{kt}$

a) $P_0 = 15$
$P(5) = 30$ find k

b) Find $P(10)$

c) Find time to reach $45 = P(t)$

Recall: $b^t = e^{\ln(b)t}$

Ex) $P(t) = 5000\,(2)^{t/4}$

$= 5000\,(2^t)^{1/4}$

$= 5000\, e^{\left(\frac{\ln 2}{4}\right)t}$

Ex) $Q(t) = Q_0 e^{-kt}$ $\quad C_{14}$ has half life 5730

Find k: Hint $Q(t) = .5Q_0$

$k \approx .000121$

Logistic Growth

Logistic Growth Model: $y = \dfrac{c}{1+ae^{-bt}}$

Ex) $P(t) = \dfrac{95.2}{1+1.8e^{-.018t}}$ $\quad$ Pop of CA.

$t = \#$ years after 2000

a) Population in 2000? $P(0) = 34$

b) Population to double? $t = 83.6$

c) Limiting value of population? 95.2 million

Section 5.1 - Systems of Linear Equations - Two Variables

Determining if a coordinate is a solution

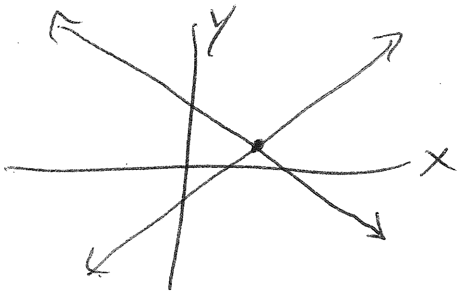
$$\begin{cases} 6x+y=-2 \\ 4x-3y=17 \end{cases}$$

(a) $(\frac{1}{2}, -5)$

(b) $(0, -2)$

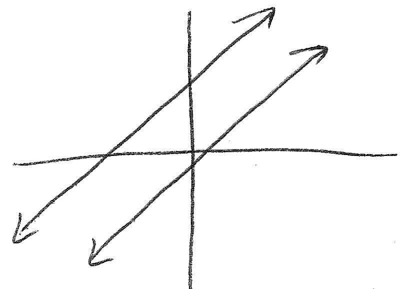
Possibilities for Solutions

One Unique Solution



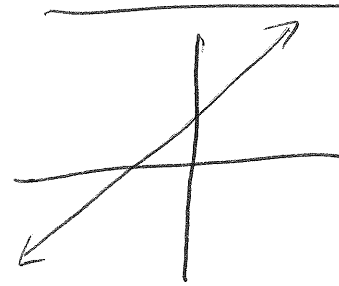
only one intersection of lines.

No Solutions



lines are parallel (inconsistent)

Infinite Solutions



formulas represent the same line (dependent)

Ex) $\begin{cases} 2x-y=3 \\ x+y=3 \end{cases}$

Ex) $\begin{cases} x+y=2 \\ x+y=-1 \end{cases}$

Ex) $\begin{cases} x-y=-1 \\ 2x-2y=-2 \end{cases}$

Methods: 1) Substitution
2) Elimination

Ex 1 $\begin{cases} -5x-4y=2 \\ 4x+y=5 \end{cases}$

Ex 2 $\begin{cases} 5x=4y+6 \\ -3x+7y=1 \end{cases}$

Ex) A boat traveling upstream against the current takes 3hrs to travel 24mi.
The return trip takes only 2hr.
Find the speed of the boat in still water.

	d	rate	t
UP	24	$b-c$	3
down	24	$b+c$	2

Ex) A lawn service has fixed cost of \$500 per month and variable cost of \$40 per lawn. If the service charges \$60 per lawn

(a) Find $C(x)$

(b) Find $R(x)$

(c) Determine break even number of lawns.

Section 5.2 - System of Linear Equations - Three Variables

Determine if (a) $(3,-5,1)$, (b) $(2,-3,1)$ are solutions to

$$\begin{cases} 2x+y-3z=-2 \\ x-4y+z=24 \\ -3x-y+4z=0 \end{cases}$$

Ex) $\begin{cases} 3x-2y+z=2 \\ 5x+y-2z=1 \\ 4x-3y+3z=7 \end{cases}$

Sol. $x=1,\ y=2,\ z=3$

Ex) $\begin{cases} 2x+y \quad =-2 \\ 3y-5z=-12 \\ 5x \quad +3z=5 \end{cases}$

Graphically: Show diagram in the book; planes intersecting

Ex) $\begin{cases} -x+6y-3z=-8 \\ x-2y+2z=3 \\ 3x+2y+4z=-6 \end{cases}$

Sol. inconsistent

Solution set notation $\{\ \}$

Ex) $\begin{cases} 2x+y \quad =-3 \\ 2y+16z=-10 \\ -7x-3y+4z=8 \end{cases}$

Sol. infinitely many

$\{(x, -2x-3, \frac{x-1}{4}) \mid x \in \mathbb{R}\}$

or $\begin{cases} x=x \\ y=-2x-3 \\ z=\frac{x-1}{4} \end{cases}$

Ex) Application

Given $(4,2)$, $(1,-1)$ and $(-1,7)$ find an equation $y = ax^2 + bx + c$ that goes through the points.

$$\left.\begin{aligned} (4,2) &\Rightarrow & 16a + 4b + c &= 2 \\ (1,-1) &\Rightarrow & a + b + c &= -1 \\ (-1,7) &\Rightarrow & a - b + c &= 7 \end{aligned}\right\} \Rightarrow \begin{aligned} a &= 1 \\ b &= -4 \\ c &= 2 \end{aligned}$$

$$\Rightarrow y = x^2 - 4x + 2$$

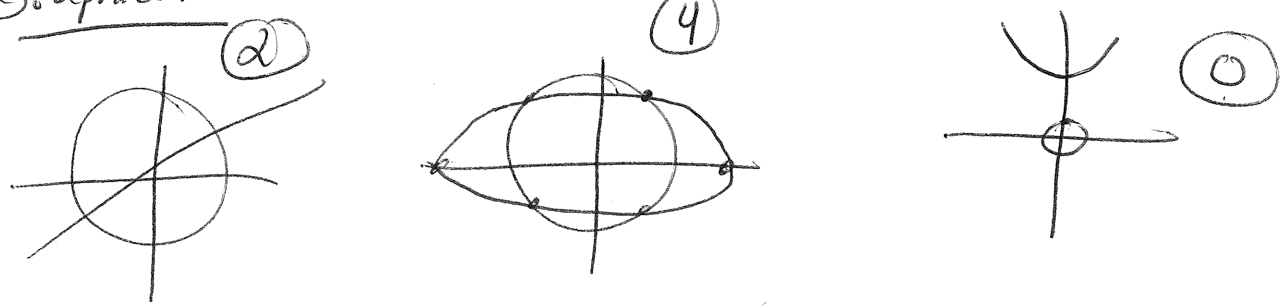
(Quadratic regression)

Section 5.4 - System of Nolinear Inequalities

Equations are non-linear, so we can have exponents otler than 1,

System can have 0, 1, 2, ..., infinitely many solutions

Graphically



Ex) $\begin{cases} -x-7y=50 \\ x^2+y^2=100 \end{cases}$

Ex) $\begin{cases} (x-5)^2+y^2=25 \\ y=3\sqrt{x-8} \end{cases}$

consider domain/Range

Ex) $\begin{cases} 2x^2+y^2=17 \\ x^2+2y^2=22 \end{cases}$

Ex) $\begin{cases} x^2+4y^2=4 \\ x^2-y^2=9 \end{cases}$

Application: A perimeter of a television is 140 in
Area is 1200 in^2

(a) find l, w
(b) Find length of diagonal

$\begin{cases} 2x+2y=140 \\ xy=1200 \end{cases}$

$x=40, 30$
$y=30, 40$

(b) $d=\pm 50$

Section 5.5 - Two Variable Systems / Inequalities

Linear Inequalities in Two Variables

Ex) $3x-2y<6$

$4x-y>3$

Ex) $2x-4y\leq 6$

$5x-15y\geq 30$

Rules

---- for $<$ or $>$

—— for $\leq$ or $\geq$

Always test to shade.

System of Linear Inequalities in 2 vars

Ex) $\begin{cases} 4y \leq 3x \\ 2y \geq 5x \end{cases}$

Ex) $\begin{cases} y \leq \frac{1}{2}x+2 \\ 3x-y<2 \end{cases}$

Ex) $\begin{cases} y < -\frac{1}{3}x+1 \\ 2x+y\leq 1 \end{cases}$

Ex) $\begin{cases} x\geq 0 \\ y\geq 0 \\ x+2y\leq 6 \end{cases}$

Ex) $\begin{cases} x\leq 1 \\ y\leq 2 \\ 2x-y\leq 4 \end{cases}$

Nonlinear Inequalities and Systems

Ex) $(x-2)^2+y^2>9$

$x^2+(y+1)^2<16$

$y^2+y-6<x$

Ex) $\frac{(x-3)^2}{4}+\frac{(x-1)^2}{9}>1$

$\frac{(x+2)^2}{4}-\frac{(x+1)^2}{9}>1$

3

Ex) $\begin{cases} x^2+y^2\leq 4 \\ y-x^2\geq 0 \end{cases}$

2

Ex) $\begin{cases} x^2+y^2\leq 25 \\ -x+y<1 \end{cases}$

5

Ex) $\begin{cases} x^2+y^2\geq 4 \\ x^2+y^2<25 \end{cases}$

4

Ex) $\begin{cases} \frac{(x)^2}{4}+\frac{y^2}{16}>1 \\ x+y<2 \end{cases}$

1

Ex) $\begin{cases} \frac{x^2}{1}-\frac{y^2}{1}>1 \\ x+y<1 \end{cases}$

Ex) $\begin{cases} x^2+y^2>16 \\ \frac{x^2}{9}+\frac{y^2}{9}<16 \end{cases}$

Ex) $\begin{cases} x^2+y^2\geq 4 \\ y-2x\geq 0 \end{cases}$

Section 5.6 – Linear Programming

p. 577, Ex 1
p. 578, Ex 2
p. 580, Ex 3
p. 581-2, Ex 4

Section 6.1 - Systems and Matrices

System of equations

$3x + 2y = 5$
$x - y + 3z = 1$
$2x + y + z = 4$

$\xRightarrow{\text{Matrix form}}$ $\left[\begin{array}{ccc|c} 3 & 2 & 0 & 5 \\ 1 & -1 & 3 & 1 \\ 2 & 1 & 1 & 4 \end{array}\right]$

① Write $2x = 5y + 5$, $3(x+y) = 17 + y$ as a augmented matrix

② Write ~~system~~ Matrix $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & 4 \end{array}\right]$ as a system.

Elementary Row Operations:
- Interchange Rows $R_1 \Leftrightarrow R_2$
- Constant Multiple Multiplication $3R_1 \Rightarrow R_1$
- Row Multiple/Addition $3R_1 + R_2 \to R_2$

Row reduce:

$\left[\begin{array}{ccc|c} 4 & 3 & 0 & 5 \\ 1 & 4 & -1 & 9 \\ 2 & 0 & -3 & -8 \end{array}\right]$

~~Gaussian Elimination~~ Rules & Defns

Row echelon form: 1 on diagonal

$\begin{bmatrix} 1 & & * \\ & 1 & \\ 0 & & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & & * \\ 0 & 1 & \\ 0 & 0 & 0 \; 0 \; 0 \end{bmatrix}$

Reduced Row Echelon $\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \; 0 \; 0 \end{bmatrix}$

Put the following into Row Echelon and Reduced Echelon form

Ex) $\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & 4 \\ 2 & -5 & 5 & 17 \end{array}\right]$

Ex) $\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}$

Ex) $\begin{bmatrix} 4 & 3 & 7 \\ 1 & 1 & 5 \\ 4 & 5 & 7 \end{bmatrix}$

Gaussian Elimination

① Put the matrix into row echelon form
② Back substitute to find the variables

Ex) $\begin{cases} 3x+7y-15z=-12 \\ x+2y-4z=-3 \\ -4x-6y+15z=16 \end{cases}$

$(-1,3,2)$

Ex) $\begin{cases} 2x+7y+z=14 \\ x+3y-z=2 \\ x+7y+12z=45 \end{cases}$

Gauss-Jordan Elimination

Steps ① ~~and~~ holds from above
② Do matrix operations in reverse to get identity matrix

Ex) $\begin{cases} 2x-5y-21z=39 \\ x-3y-10z=22 \\ x+3y+2z=-8 \end{cases}$

Ex) $\begin{cases} 2x+7y+11z=11 \\ x+2y+8z=14 \\ x+3y+6z=8 \end{cases}$

Ex) $\begin{cases} x-2y=-1 \\ 4x-7y=1 \end{cases}$

Section 6.2 - Inconsistent / Dependent Systems

Ex) Solve (Inconsistent)

$$\begin{cases} x-3y-17z=-59 \\ x-2y-12z=-41 \\ -2y-10z=20 \end{cases}$$

$$\begin{cases} 5x-9y-33z=3 \\ x-2y-7z=0 \\ -2x+y+8z=-12 \end{cases}$$

Ex) Solve (Dependent)

$$\begin{cases} x-4y+7z=14 \\ -2x+9y-16z=-31 \\ x-7y+13z=23 \end{cases}$$

$$\begin{cases} -4x-11y+3z=-24 \\ x+3y-z=7 \\ 3x+11y-5z=29 \end{cases}$$

Systems where # Eqns < # Variables

Ex) $\begin{cases} 3x-8y+18z=15 \\ x-3y+4z=6 \end{cases}$

* Contradiction ⇒ parallel planes
* Identity ⇒ 2 eqns rep same plane
* Otherwise ⇒ planes meet on a line

Ex) $\begin{cases} x-3y-17z=-17 \\ -2x+7y+38z=40 \end{cases}$

Solving a dependent system.

Ex) $\begin{cases} x+2y+3z=6 \\ -x-2y-3z=-6 \\ 2x+4y+6z=12 \end{cases}$

Ex) $\begin{cases} 2x-3y+5z=3 \\ 4x-6y+10z=6 \\ 20x-30y+50z=30 \end{cases}$

Word Problems

① John inherits \$25,000 and invests in

$x=$ MMA @ 6%
$y=$ MBonds @ 7% annually
$z=$ MFund @ 8%

After 1 year, \$1620 from all investments. If \$6000 more was invested in bonds than mutual funds. How much invested in each

$$\Rightarrow \begin{aligned} x+y+z &= 25{,}000 \\ .06x+.07y+.08z &= 1620 \\ y-z &= 6{,}000 \end{aligned} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ .06 & .07 & .08 \\ 0 & 1 & -1 \end{bmatrix}$$

②

Section 6.2 (cont.)

Systems with fewer equations than unknowns.

$\Rightarrow$ more columns than rows

In 3D, each equation is a plane in space

- Contradiction $\Rightarrow$ no intersection, planes are parallel
- Identity $\Rightarrow$ equations represent same plane
- Otherwise $\Rightarrow$ planes meet on a common line.

Ex) $\begin{cases} 3x - 8y + 18z = 15 \\ x - 3y + 4z = 6 \end{cases}$ Ex) $\begin{cases} x - 3y - 17z = -17 \\ -2x + 7y + 38z = 40 \end{cases}$

Dependent System $\Rightarrow$ "free variables" recall from last lecture

Possible to have more than 1 free variable

Ex) $\begin{cases} x + 2y + 3z = 6 \\ -x - 2y - 3z = -6 \\ 2x + 4y + 6z = 12 \end{cases}$ Ex) $\begin{cases} 2x - 3y + 5z = 3 \\ 4x - 6y + 10z = 6 \\ 20x - 30y + 50z = 30 \end{cases}$

Word Problem John invests an inheritance of \$25,000 in x = MMA @ 6%, y = M bonds @ 7%, z = M funds @ 8% annually. After 1 yr, \$1620 is ~~made~~ gained from the investments. If \$6000 more was invested in bonds than M funds, How much invested in each?

$$\begin{cases} x+y+z = 25{,}000 \\ .06x + .07y + .08z = 1620 \\ y - z = 6{,}000 \end{cases}$$

$-.06(25000) = -1500$

$.01 = \frac{1}{100}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 25{,}000 \\ .06 & .07 & .08 & 1{,}620 \\ 0 & 1 & -1 & 6{,}000 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 25{,}000 \\ 0 & .01 & .02 & 120 \\ 0 & 1 & -1 & 6{,}000 \end{array}\right]$$

$$\xrightarrow{100R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 25{,}000 \\ 0 & 1 & 2 & 12{,}000 \\ 0 & 1 & -1 & 6{,}000 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 25{,}000 \\ 0 & 1 & 2 & 12{,}000 \\ 0 & 0 & -3 & -6{,}000 \end{array}\right] \xrightarrow{-\frac{1}{3}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 25{,}000 \\ 0 & 1 & 2 & 12{,}000 \\ 0 & 0 & 1 & 2{,}000 \end{array}\right]$$

$\Longrightarrow \boxed{z = 2{,}000}$ $\quad$ (2) $\Rightarrow$ $y + 2z = 12{,}000$

$\boxed{y = 8{,}000}$

$\boxed{x = 15{,}000}$

Section 6.3

Recall, $M \times N$ matrix $\Rightarrow$ M rows, N columns.

* Examples on sizes.

* Notation $[a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$i \Rightarrow$ row, $j \Rightarrow$ column

* Identifying elements using i,j notation

* Adding/Subtracting Rules. Dimensions must agree

* Examples A, B, C, then various combos

Section 6.4 - Inverse Matrices

Defn: I_n is the $n \times n$ identity matrix, which has 1's on main diagonal, 0's everywhere else

Properties: $A \cdot I_n = A$, $I_n A = A$

Ex) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and vice-versa.

Defn: Let A be $n \times n$ matrix. If there exists an $n \times n$ matrix, A^{-1}, s.t.

$$AA^{-1} = I_n \quad \text{and} \quad A^{-1}A = I_n$$

then A^{-1} is the inverse of A.

Ex) Are $A = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$ inverses?

Find inverses A^{-1}

Ex) $A = \begin{bmatrix} 2 & 9 \\ 1 & 5 \end{bmatrix}$ Ex) $A = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & -2 \\ -3 & -3 & 5 \end{bmatrix}$

Ex) $A = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 5 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ Ex) $\begin{bmatrix} 2 & 9 \\ 1 & 5 \end{bmatrix}$ determinant

(singular)

Ex) Solve $\begin{cases} x + 3y = -10 \\ x + y - 2z = -4 \\ -3x - 3y + 5z = 11 \end{cases}$ Using inverse matrix

$A\vec{x} = \vec{b}$

$(A^{-1}A)\vec{x} = A^{-1}\vec{b}$

$I_n \vec{x} = A^{-1}\vec{b}$

$\vec{x} = A^{-1}\vec{b}$

Section 6-5 – Determinants and Cramers' Rule

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Examples: Any, and $A = \begin{pmatrix} 2 & -11 \\ 0 & 0 \end{pmatrix}$

2×2 examps

3×3 examples (1 @ pivot)

Cofactor Matrix: for $A = [a_{ij}]$, the cofactor of a_{ij} is $(-1)^{i+j} M_{ij}$, M is the minor

$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ = cofacter matrix

Determine if $A = \begin{bmatrix} -2 & 0 & 1 & 2 \\ 6 & 2 & -2 & 5 \\ 5 & 3 & -1 & 1 \\ 0 & 4 & 2 & 7 \end{bmatrix}$

$= -2(1)(42) + 0 + 1(1)(84) + 2(-1)(0) = 0$

$\Rightarrow$ not invertible

A matrix is singular if $\det(A) = 0$, i.e. not invertible.

Section 2.1 - Circles

Defn - A circle is a set of equidistant points from a fixed point called center; The distance is r

Standard Form: $(x-h)^2+(y-k)^2=r^2$ (write in ellipse form)

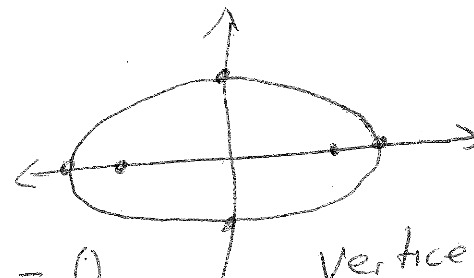
General Form: $x^2+y^2+Ax+By+C=0$

Ex) Write eqn of circle with endpoints of a diameter $(-1,0),(3,4)$

Ex) Write $x^2+y^2+10x-6y+25=0$ in standard form
Write $x^2+y^2-8x+2y-8=0$ " " "

Graph Examples above

Section 7.1 - Ellipse



Standard Form: $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

General Form: $Ax^2+By^2+Cx+Dy+E=0$
$A,B>0$

Vertices
foci
major/minor axes

Foci Equation: $c^2=|a^2-b^2|$ $(a>b>0)$

Ex) Graph, find quantities: $\frac{x^2}{16}+\frac{y^2}{9}=1$

Test Question $\frac{(x-3)^2}{16}+\frac{(y+2)^2}{25}=1$
" "

Foci $\rightarrow (h \pm c, k \pm c)$

Ex) $3x^2+7y^2+6x-28y+10=0$

$2x^2+11y^2-12x+44y+90=0$

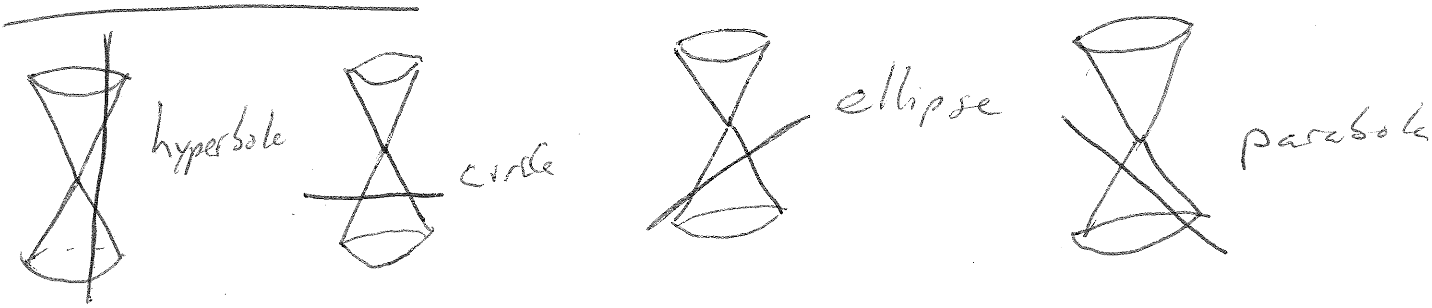
Ex) Determine Standard Form with vertices
$(4,2), (-4,2)$; foci $(\sqrt{15},2), (-\sqrt{15},2)$

Ex) Eccentricity

$e=\frac{c}{a}$ where $a>b>0$, $c>0$
$c^2=a^2-b^2$

$0<e<1$

Section 7.2 – Hyperbolas



Ex) Graph $\frac{x^2}{4} - \frac{y^2}{9} = 1$, $\frac{x^2}{9} - \frac{y^2}{4} = 1$

Standard Form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ or $-\frac{(x-h)^2}{a^2} + \frac{(y-h)^2}{b^2} = 1$

General Form: $Ax^2 + By^2 + Cx + Dy + E = 0$

$A > 0 > B$ or $B > 0 > A$

Center: (h, k)

Foci $c^2 = a^2 + b^2$

vertices

Asymptotes: use graph!

Ex) Graph $-\frac{(x-2)^2}{9} + (y+3)^2 = 1$

Ex) Complete the square: $11x^2 - 2y^2 + 66x - 4y + 75 = 0$

$3x^2 - 7y^2 + 30x + 56y - 58 = 0$

Ex) Determine hyperbola given vertices $(-4,0), (8,0)$

foci $(-8,0), (12,0)$

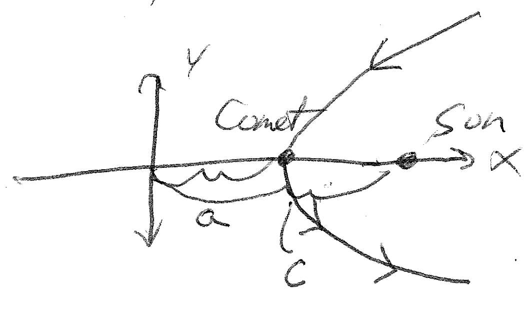
Eccentricity $e = \frac{c}{a}$ $c^2 = a^2 + b^2$

Higher the eccentricity, hyperbola opens wider

Application: Comets can have hyperbolic paths $\Rightarrow$ comet not captured

Comet C/2009 R_1 has path described by

$$\frac{x^2}{(1191.2)^2} - \frac{y^2}{(30.9)^2} = 1$$



(a) determine the distance in AU at perihelion

(b) If $1\,AU = 9.3 \times 10^{7}$ mi, find dist in mi

$$a^2 = (1191.2)^2 \qquad \Rightarrow \quad c^2 = a^2 + b^2$$
$$b^2 = (30.9)^2 \qquad\qquad c^2 = 1{,}419{,}912.25$$
$$\Rightarrow c = \pm 1{,}191.6$$

$$d = c - a = 1191.6 - 1191.2 = 0.4\,AU$$

(b) $1\,AU = 9.3 \times 10^{7}$ mi

$$d = 0.4\,\cancel{AU} \cdot \frac{9.3 \times 10^{7}\text{ mi}}{1\,\cancel{AU}} = 37.2 \text{ million miles}$$

Section 7.3 - Parabolas

Standard Form: $(x-h)^2 = 4p(y-k)$
$(y-k)^2 = 4p(x-h)$

vertex $= (h,k)$

	x^2	y^2
foci	$(h, k+p)$	$(h+p, k)$
directrix	$y = k-p$	$x = h-p$
A.O.S.	$x = h$	$y = k$
	$p>0$ open up, $p<0$ open down	$p>0$ open right, $p<0$ open left

Defn: A parabola is a set of points that are equidistant from a fixed line ($\Rightarrow$ directrix) and fixed point (focus), focal diameter $= 4|p|$

Distance between focus and vertex is the focal length.

Ex) Graph $x^2 = -12y$, identify vertex, focus, directrix, a.o.s

$2y^2 = 32x \rightarrow y^2 = 16x$

$(y+3)^2 = -4(x-2)$

Ex) $4x^2 - 20x - 8y + 57 = 0$
$4x^2 - 12x - 12y + 21 = 0$

Ex) Determine parabola with focus $(1, 2)$, directrix $x = 7$

$(y-2)^2 = -12(x-4)$

Section 8.2 - Arithmetic Sequences and Series

Defn: An arithmetic sequence $\{a_n\}$ is a sequence in which each term after the first differs from the previous by a fixed constant, d, called common ~~sequence~~ difference computed by $d = a_{n+1} - a_n$

Ex) Determine if the sequence is arithmetic

(a) $12, 5, -2, -9, -16, \ldots$

(b) $1, 4, 9, 16, 25, \ldots$

Ex) Write first 5 terms of arithmetic sequence with first term -5 and common difference 4.

$$a_1 = -5, \; d = 4$$

$$\{-5, -1, 3, 7, 11\}$$

Looking at construction of sequence

$a_1 = a_1$ $\quad a_3 = a_2 + d = a_1 + 2d$ $\quad \ldots \quad a_n = a_1 + (n-1)d$

$a_2 = a_1 + d$ $\quad a_4 = a_3 + d = a_1 + 3d$

Ex) Find the 9th term of the arithmetic seq. which has $a_1 = -4$, $a_{22} = 164$

$a_n = a_1 + (n-1)d$ $\qquad a_n = -4 + (n-1)8$

$164 = -4 + (22-1)d$ $\qquad \Rightarrow a_9 = 60$

Ex) $a_1 = 12, \; a_{30} = 128$

$$S_n = a_1 + (a_1+d) + (a_1+2d) + \dots + a_n$$
$$+\ \underline{S_n = a_n + (a_n-d) + (a_n-2d) + \dots + a_1}$$

$$2S_n = (a_1+a_n) + (a_1+a_n) + \dots + (a_1+a_n)$$
$$2S_n = n(a_1+a_n)$$
$$\boxed{S_n = \frac{n}{2}(a_1+a_n)}$$

n^{th} Partial sum arithmetic sequence

Ex) Find the first 50 terms of 2, 4, 6, 8, 10, ...
S_{50}

Ex) Find $\sum_{i=1}^{60}(3i+5)$, $\sum_{i=1}^{80}(4i+3)$

8.3 - Geo series/seq
8.4 - Induction
8.5 - Binomial Thm

Ex) $\sum_{i=1}^{n} i$, $\sum_{i=1}^{n} i^2$

Section 8.3 - Geometric Seq. and Series

Defn. A geometric sequence is a sequence that is constructed by multiplying the previous term by one fixed number (non-zero) called the common ratio.

We can write this in general form as

$\{a, ar, ar^2, ar^3, ar^4, \ldots\}$ where a is the first term, r is the common ratio

In our usual notation, we have $a_n = ar^{n-1}$ $n \geq 1$

and recursively, $a_n = r\,a_{n-1}$ for $n \geq 1$

The common ratio can be negative, for example

$1, -2, 4, -8, 16, -32, 64, \ldots$ here $r = -2$

Ex) Find general term for a) $2, 8, 32, 128, \ldots$

b) $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \ldots$

Ex) Find the 10^{th} term of geo. series if $a_1 = 45$, $r = \frac{1}{5}$

$a_n = ar^{n-1} \Rightarrow a_{10} = 45\left(\frac{1}{5}\right)^{10-1} = 2.304 \times 10^{-5}$

Ex) Find the 20^{th} term of geo series if $a_1 = -\frac{1}{2}$, $r = -\frac{1}{2}$

$a_{20} = -\frac{1}{2}\left(-\frac{1}{2}\right)^{20-1} = -\frac{1}{2}\left(-\frac{1}{2}\right)^{19} = \left(-\frac{1}{2}\right)^{20} = \boxed{\frac{1}{2^{20}}}$

Defn: A <u>geometric series</u> is the sum of the terms in a geometric sequence.

We know that $a_n = ar^{n-1}$

$$\Rightarrow \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} ar^{k-1} = a + ar^1 + ar^2 + \dots + ar^{n-1}$$

$$\Rightarrow (1-r)\sum_{k=1}^{n} ar^{k-1} = \begin{aligned} &a + ar^1 + ar^2 + \dots + ar^{n-1} \\ - &ar - ar^2 - ar^3 - \dots - ar^n \end{aligned}$$

$$= a - ar^n = a(1-r^n)$$

$$\Rightarrow \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} ar^{k-1} = \boxed{\frac{a(1-r^n)}{(1-r)} = S_n}$$

$$\sum_{k=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

Ex) Find $\sum_{k=1}^{6} 3^{k-1}$, $\sum_{k=1}^{10} 8\left(\frac{1}{4}\right)^{k-1}$ <u>Sol</u>: 10.67

Ex) $\sum_{i=1}^{\infty} 4\left(\frac{3}{4}\right)^{i-1}$