MATH 65B FINAL July 27, 2016

Name: KEY Score: ________ / 100

Student ID: ______________________

DO NOT OPEN THE EXAM UNTIL YOU ARE TOLD TO DO SO

	1	2	3	4	5	6	7	8	9	Total
✓										70
Score										
Pts. Possible	6	8	8	8	8	8	12	12	12	82

INSTRUCTIONS FOR STUDENTS

- Questions are on both sides of the paper. This is an 9 question exam.
- Students have 2 hours and 15 minutes to complete the exam.
- The test will be out of **70 points**. The highest possible score will be **82 points**. You can attempt as many of the questions as you wish, but keep in mind you are trying to get to the **70 points**.
- In the above table, the row with the ✓, is for you to keep track of the problems you are attempting/completing.
- Higher point problems are harder, thus they are weighted more. In order to do well, you will have to attempt some of the more difficult problems.
- You may complete parts of problems, as partial credit will be given based on correctness, completeness, and ideas that are leading to the correct solutions.
- **PLEASE SHOW ALL WORK**. Any unjustified claims will receive no credit. This means you need to state which test you are using for series questions! Clearly box your final answer.
- No notes, textbooks, phones, calculators, etc. are allowed for the exam.
- The back of the test can be used for scratch work.

GOOD LUCK!

FORMULAS:

Common Taylor Series	Common Taylor Series
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, for all $\lvert x\rvert < 1$	$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, for all $x \in \mathbb{R}$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, for all $x \in \mathbb{R}$	$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, for all $x \in \mathbb{R}$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$, for $x \in (-1, 1]$	$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$, for $\lvert x\rvert \le 1$
$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, for $\lvert x-a\rvert < R$	$(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n$, for $\lvert x\rvert < 1$

1) (3 pts.) (a) Determine whether the sequence converges or diverges:

$$a_n = \frac{\cos^2(n)}{2^n}$$

(3 pts.) (b) Determine whether the series converges or diverges. If it converges, find its sum:

$$\sum_{n=1}^{\infty} \arctan(n)$$

(a) We have $0 \leq \frac{\cos^2(n)}{2^n}$ since $\cos^2(n), 2^n > 0$ for all n

$|\cos^2(n)| \leq 1$ so then

$0 \leq \frac{\cos^2(n)}{2^n} \leq \frac{1}{2^n}$, since $\lim_{n\to\infty} \frac{1}{2^n} = 0$

by squeeze theorem, a_n converges

(b) $\lim_{n\to\infty} \arctan(n) = \frac{\pi}{2} \neq 0$ so by

Test for Divergence, Series diverges.

2) (6 pts.) Determine whether the series is convergent or divergent

$$\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$$

Compare with $f(x) = \dfrac{1}{x\ln(x)}$

① f is cont. on $[2,\infty)$ as its only undefined at $\cancel{x=0}$ $x=1$

② f is positive as $x>0$, $\ln(x)>0$ for $x>1$

③ $f'(x) = -\dfrac{1+\ln(x)}{(x\ln(x))^2} < 0$ for $x>2$ $\Rightarrow$ decreasing

$$\Rightarrow \int_2^\infty \frac{1}{x\ln(x)}\,dx = \lim_{t\to\infty} \ln(\ln(x))\Big|_2^t$$
$$= \lim_{t\to\infty} \ln(\ln(t)) - \ln(\ln(2))$$
$$= \infty$$

$\Rightarrow$ divergent by Integral Test.

3) (8 pts.) Determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$$

Limit Comparison Test

$$b_n = \frac{\sqrt{n}}{n^2} = \frac{n^{1/2}}{n^2} = \frac{1}{n^{3/2}} \Rightarrow \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$ convergent p-series

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{n^2+1} \cdot \frac{n^{3/2}}{1} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 < \infty$$

$\Rightarrow$ By Limit comparison test,

$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ Converges since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ Converges

4) (4 pts.) (a) Determine whether the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$$

(4 pts.) (b) Determine whether the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$$

(a) Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)\cancel{n!}}{\cancel{n!}} \cdot \frac{e^{n^2}}{e^{n^2+2n+1}}$$

$$= \lim_{n\to\infty} \frac{n+1}{e^{2n+1}} = 0 < 1 \Rightarrow$$

$\boxed{\text{(absolutely) Convergent by Ratio Test}}$

(b) Root Test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)^{n^2}\right)^{1/n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$= e > 1$$

$\boxed{\text{By Root Test, series diverges}}$

5) (8 pts.) Determine whether the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n^2-1}}$$

① Check absolute Convergence

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\sqrt{n^2-1}} \right| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2+1}}$$

Limit Comparison with $\frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} \cdot \frac{n}{1} \stackrel{L'H}{=} \lim_{n\to\infty} \frac{n}{\sqrt{n^2}} = 1 > 0$$

$\Rightarrow$ divergent since compared with $\sum_{n=2}^{\infty} \frac{1}{n}$ (divergent)

Does not converge absolutely.

② Check Alternating Series Test

ⓐ $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n^2-1}} = 0$ ✓

ⓑ $\sqrt{n^2 \mp 1} < \sqrt{(n+1)^2 + 1} \Rightarrow \frac{1}{\sqrt{(n+1)^2-1}} < \frac{1}{\sqrt{n^2-1}}$ ✓

$\Rightarrow$ decreasing

So convergent by Alternating Series Test

$$\Rightarrow \boxed{\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n^2-1}} \text{ is conditionally convergent}}$$

6) (8 pts.) Find the radius of convergence and interval of convergence for the following power series (This is known as the *Bessel function of order 1*):

$$\sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+1}}{n! \, (n+1)! \, 2^{2n+1}}$$

Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{x^{2n+3}}{(n+1)!\,(n+2)!\,2^{2n+3}} \cdot \frac{n!\,(n+1)!\,2^{2n+1}}{x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^{2n+3}}{x^{2n+1}} \right| \cdot \frac{2^{2n+1}}{2^{2n+3}} \cdot \frac{\cancel{n!\,(n+1)!}}{(n+2)(n+1)\cancel{n!\,(n+1)!}}$$

$$= \frac{|x|^2}{2^2} \lim_{n\to\infty} \frac{1}{(n+2)(n+1)} = 0 \text{ for } \underline{\underline{\text{all}}}\ x$$

$\Rightarrow R = \infty$, Interval of convergence is $(-\infty, \infty)$

7) (6 pts.) (a) Compute the following integral using Taylor series. State the radius of convergence. (*Hint: Be careful about the $n = 0$ term, you can't have 0/0.*)

$$\int \frac{e^x}{x}\, dx$$

(6 pts.) (b) Find the Taylor series centered at $a = 0$, and state the radius of convergence.

$$f(x) = \ln(x)$$

(a) We know from the table on the front (line 1 below)

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
\frac{1}{x} e^x &= \sum_{n=0}^{\infty} \frac{x^{n-1}}{n!} \\
&= \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} \\
\int \frac{e^x}{x}\, dx &= \int \frac{1}{x}\, dx + \sum_{n=1}^{\infty} \int \frac{x^{n-1}}{n!}\, dx \\
&= \ln|x| + \sum_{n=1}^{\infty} \int \frac{x^n}{n \cdot n!} + C
\end{aligned}$$

The radius of convergence in infinity since it is infinity for e^x by Theorem.

(b) We know from the table on the front (line 1 below)

$$\begin{aligned}
\ln(1+x) &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \\
\ln(1+(x-1)) &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n} \\
\ln(x) &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}
\end{aligned}$$

8) (6 pts.) (a) Eliminate the parameter in the for the following parametric equation

$$x = 4\cos(t) + 2 \quad y = 2\sin(t) + 1$$

(2 pts.) (b) Identify the type of graph from your result in part (a).

(4 pts.) (c) Compute $\frac{dy}{dx}$. What coordinates have a tangent line with slope 0?

(a) $x = 4\cos(t) + 2 \qquad y = 2\sin(t) + 1$

$(x-2)^2 = 16\cos^2(t) \qquad (y-1)^2 = 4\sin^2(t)$

$\frac{(x-2)^2}{16} = \cos^2(t) \qquad \frac{(y-1)^2}{4} = \sin^2(t)$

$$\Rightarrow \boxed{\frac{(x-2)^2}{16} + \frac{(y-1)^2}{4} = 1}$$

(b) Ellipse center at $(2,1)$ $a = 4, b = 2$

(c) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos(t)}{-4\sin(t)} = -\frac{1}{2} \cdot \frac{\cos(t)}{\sin(t)}$

$\frac{dy}{dx} = 0 = -\frac{1}{2}\frac{\cos(t)}{\sin(t)} \Leftrightarrow \cos(t) = 0$

$\Rightarrow t = (n - \frac{1}{2})\pi \quad n \in \mathbb{Z}$

Recall: $\left.\begin{array}{l} \cos(n-\frac{1}{2})\pi = 0 \\ \sin(n-\frac{1}{2})\pi = (-1) \end{array}\right\}$

$x = 4\cos(n-\frac{1}{2})\pi + 2$

$x = 2$

$y = 2\sin(n-\frac{1}{2})\pi + 1$

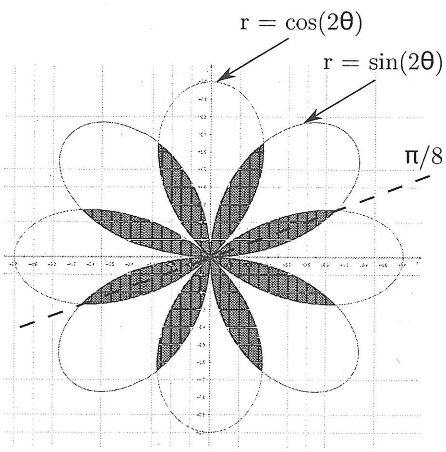
$y = 3, -1$

$\Rightarrow \boxed{\begin{array}{c}(2,3)\\(2,-1)\end{array}}$

9) (12 pts.) Find the area of the region that lies inside the polar roses (the region is shaded in the labeled plot below).

Hint 1: Use the symmetry at $\frac{\pi}{8}$ *to simplify the integral. How many pieces do you really have?*
Hint 2: Use the identity: $\sin^2(2\theta) = \frac{1}{2} - \frac{1}{2}\cos(4\theta)$.

$$r = \cos(2\theta)$$
$$r = \sin(2\theta)$$



"Find the intersection"

$\sin(2\theta) = \cos(2\theta)$.

$\Rightarrow \dfrac{\sin(2\theta)}{\cos(2\theta)} = \tan(2\theta) = 1$

$\Rightarrow 2\theta = \frac{\pi}{4}$

$\Rightarrow \theta = \pi/8$

$$A = \frac{1}{2}\int_a^b [f(\theta)]^2\, d\theta = \frac{1}{2}\int_a^b r^2\, d\theta$$

$$A = 8\cdot 2\int_0^{\pi/8} \frac{1}{2}\sin^2(2\theta)\, d\theta$$

$$A = 8\int_0^{\pi/8} \left(\frac{1}{2} - \frac{1}{2}\cos(4\theta)\right) d\theta$$

$$A = 4\left(\theta - \frac{1}{4}\sin(4\theta)\right)\Big|_0^{\pi/8}$$

$$\boxed{A = \frac{\pi}{2} - 1}$$

THIS PAGE IS LEFT BLANK FOR ANY SCRATCH WORK

END OF TEST