MATH 65B MIDTERM June 30, 2016

Name: KEY Score: ________ / 100

Student ID: ______________________________

DO NOT OPEN THE EXAM UNTIL YOU ARE TOLD TO DO SO

	1	2	3	4	5	6	7	8	9	Total
✓										50
Score										
Pts. Possible	~~8~~ 10	6	3	5	~~6~~ 8	6	8	6	6	~~54~~ 56

INSTRUCTIONS FOR STUDENTS

- Questions are on both sides of the paper. This is an 9 question exam.
- Students have 2 hours and 15 minutes to complete the exam.
- The test will be out of **50 points**. The highest possible score will be **54 points**. You can attempt as many of the questions as you wish, but keep in mind you are trying to get to the **50 points**.
- In the above table, the row with the ✓, is for you to keep track of the problems you are attempting/completing.
- Higher point problems are harder, thus they are weighted more. In order to do well, you will have to attempt some of the more difficult problems.
- You may complete parts of problems, as partial credit will be given based on correctness, completeness, and ideas that are leading to the correct solutions.
- **PLEASE SHOW ALL WORK**. Any unjustified claims will receive no credit. Clearly box your final answer.
- No notes, textbooks, phones, calculators, etc. are allowed for the exam.
- The back of the test can be used for scratch work.

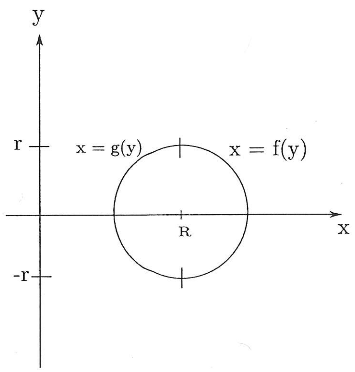
GOOD LUCK!

FORMULAS:

Useful Formulas	Useful Formulas
$\frac{d(\arcsin(x))}{dx} = \frac{1}{\sqrt{1-x^2}} \quad \lvert u\rvert < 1$	$\int \frac{dx}{\sqrt{a^2+x^2}} = \arcsin\left(\frac{x}{a}\right) + C$
$\frac{d(\arccos(x))}{dx} = -\frac{1}{\sqrt{1-x^2}} \quad \lvert u\rvert < 1$	$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$
$\frac{d(\arctan(x))}{dx} = \frac{1}{1+x^2}$	$\int \frac{dx}{u\sqrt{a^2-x^2}} = \frac{1}{a}\,\mathrm{arcsec}\left\lvert\frac{x}{a}\right\rvert + C$

1) (~~8pts.~~ 10) The following question is designed to walk you through how to find the volume of the torus, the doughnut shaped solid pictured at the bottom of the page.

a) (1pt.) Write the equation of a circle with center at $(R, 0)$ and radius r.
b) (1pt.) Solve the equation of the circle you found in part (a) for x. Now, using the result, write out the functions $g(y)$ and $f(y)$ that are given in the diagram below.
c) (3pts.) Integrating with respect to y, set up the integral for the volume. We are really doing washers from scratch. We are finding the area of the large outside disk and small inner disk. (*Hint: Since we are integrating with respect to y, the formula is no longer $top - bottom$, but $right - left$.*)
d) (3pts.) Do the integration and find the volume, using the answer from part (c).



a) $(x-R)^2 + y^2 = r^2$

b) $x = R \pm \sqrt{r^2-y^2}$

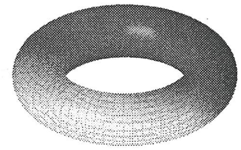
$\Rightarrow x = f(y) = R + \sqrt{r^2-y^2}$
$x = g(y) = R - \sqrt{r^2-y^2}$

c) $V = \pi \int_a^b [f(y)^2 - g(y)^2]\, dy = \pi \int_{-r}^{r} \left[\left(R+\sqrt{r^2-y^2}\right)^2 - \left(R-\sqrt{r^2-y^2}\right)^2\right] dy$

d) $= \pi \int_{-r}^{r} \left[\left(R^2 + 2R\sqrt{r^2-y^2} + r^2 - y^2\right) - \left(R^2 - 2R\sqrt{r^2-y^2} + r^2 - y^2\right)\right] dy$

$= \pi \int_{-r}^{r} 4R\sqrt{r^2-y^2}\, dy = 4\pi R \int_{-r}^{r} \sqrt{r^2-y^2}\, dy$

$= 4\pi R \cdot \frac{1}{2}\pi r^2 = \boxed{2\pi^2 r^2 R}$



2) (6pts.) The following question is designed for you to derive the surface area of a sphere.

Find the area of the surface generated by revolving the curve $y = \sqrt{R^2 - x^2}$, for $-R \leq x \leq R$ about the x-axis.

$$y = \sqrt{R^2 - x^2} \Rightarrow \frac{dy}{dx} = \frac{1}{2} \frac{-2x}{\sqrt{R^2-x^2}} = \frac{-x}{\sqrt{R^2-x^2}}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{x}{R^2-x^2} \Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{x}{R^2-x^2}$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = \frac{R^2}{R^2-x^2}$$

$$S = \int_{-R}^{R} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{-R}^{R} 2\pi \sqrt{R^2-x^2} \sqrt{\frac{R^2}{R^2-x^2}}\, dx$$

$$= \int_{-R}^{R} 2\pi R\, dx = 2\pi R \int_{-R}^{R} dx$$

$$= 2\pi R(R-(-R))$$

$$= 2\pi R(2R)$$

$$= \boxed{4\pi R^2}$$

3) (3pts.) A particle is moved along the x-axis by a force that measures $\dfrac{10}{(1+x)^2}$ pounds at a point x feet from the origin. Find the work done in moving the particle from the origin to a distance of 9 feet.

$$W = \int_a^b F(x)\,dx \qquad F(x) = \frac{10}{(1+x)^2} \quad \text{from } x=0 \text{ to } x=9$$

$$\Rightarrow W = \int_0^9 \frac{10}{(1+x)^2}\,dx = 10\int_1^{10} \frac{1}{u^2}\,du \qquad u = x+1, \quad du = dx$$

$$= 10\left(-\frac{1}{u}\Big|_1^{10}\right) = 10\left(-\frac{1}{10}+1\right)$$

$$= \boxed{9 \text{ ft-lbs}}$$

4) (3pts.) (a) Solve the initial value problem $\frac{dy}{dx} = \frac{1}{x^2+1} \cancel{-1}, \quad y(0) = 1$

(2pts.) (b) From part (a), you will have a function, $y(x)$ which satisfies the above equation. Use the answer from part (a) to compute: $\lim_{x\to\infty} y(x)$.

a) $\frac{dy}{dx} = \frac{1}{x^2+1}$

$\Rightarrow \int dy = \int \frac{1}{x^2+1}\,dx$

$\Rightarrow \boxed{y = \arctan(x) + C}$

$\Rightarrow y(0) = \arctan(0) + C$

$1 = C$

$\Rightarrow \boxed{y = \arctan(x) + 1}$

b) $\lim_{x\to\infty} \left(\arctan(x) + 1\right)$ $\qquad \lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$

$= \boxed{\frac{\pi}{2} + 1}$

(8 pts)

5) (~~6pts.~~) Compute the following integral

$$\int_0^t e^s \sin(t-s)\, ds$$

Integration by parts with respect to s

$u = \sin(t-s)$ $\qquad dv = e^s ds$

$du = -\cos(t-s) ds$ $\qquad v = e^s$

$$\Rightarrow \int_0^t e^s \sin(t-s)\, ds = e^s \sin(t-s)\Big|_{s=0}^{s=t} - \int (-\cos(t-s)) e^s\, ds$$

$$= e^s \sin(t-s)\Big|_{s=0}^{s=t} + \int_0^t e^s \cos(t-s)\, ds$$

$u = \cos(t-s)$ $\quad dv = e^s ds$

$du = \sin(t-s) ds$ $\quad v = e^s$

$$= e^s \sin(t-s)\Big|_{s=0}^{s=t} + e^s \cos(t-s)\Big|_{s=0}^{s=t} - \int_0^t e^s \sin(t-s)\, ds$$

$$\Rightarrow 2\int_0^t e^s \sin(t-s)\, ds = e^s \sin(t-s)\Big|_{s=0}^{s=t} + e^s \cos(t-s)\Big|_{s=0}^{s=t}$$

$$= e^t \sin(t-t) - e^0 \sin(t-0) + e^t \cos(t-t) - e^0 \cos(t-0)$$

$$= e^t \cancel{\sin(0)} - e^0 \sin(t) + e^t \cos(0) - e^0 \cos(t)$$

$$2\int_0^t e^s \sin(t-s) ds = -\sin(t) + e^t - \cos(t)$$

$$\Rightarrow \boxed{\int_0^t e^s \sin(t-s)\, ds = \frac{1}{2}\left(e^t - \sin(t) - \cos(t)\right)}$$

6) (6pts.) Evaluate the integral

$$\int \cos^5(x)\, dx$$

$$\int \cos^5(x)\,dx = \int \cos^4(x)\cos(x)\,dx$$

$$= \int (1-\sin^2(x))^2 \cos(x)\,dx$$

$u = \sin(x)$
$du = \cos(x)\,dx$

$$= \int (1-u^2)^2\,du$$

$$= \int (u^4 - 2u^2 + 1)\,du$$

$$= \frac{u^5}{5} - \frac{2}{3}u^3 + u + C$$

$$= \boxed{\frac{\sin^5(x)}{5} - \frac{2}{3}\sin^3(x) + \sin(x) + C}$$

7) (8pts.) Evaluate the integral

$$\int \frac{\sqrt{x^2-9}}{x^3}\,dx$$

Let $x = 3\sec\theta \quad dx = 3\sec\theta\tan\theta\,d\theta$

$$\Rightarrow \int \frac{\sqrt{x^2-9}}{x^3}\,dx = \int \frac{\sqrt{9\sec^2\theta - 9}}{27\sec^3\theta}\, 3\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{3\tan\theta}{27\sec^3\theta}\cdot 3\sec\theta\tan\theta\,d\theta$$

$3\sqrt{\tan^2\theta}$

$= 3\tan\theta$

$$= \frac{1}{3}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta = \frac{1}{3}\int \sin^2\theta\,d\theta$$

$x = 3\sec\theta$

$\Rightarrow \frac{x}{3} = \sec\theta$

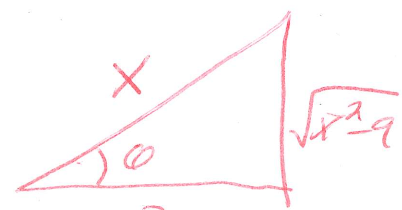
$\theta = \sec^{-1}\left(\frac{x}{3}\right)$

$\sin\theta = \frac{\sqrt{x^2-9}}{x}$

$\cos\theta = \frac{3}{x}$

$$= \frac{1}{6}\int (1-\cos 2\theta)\,d\theta$$

$$= \frac{1}{6}\left[\theta - \frac{1}{2}\sin(2\theta)\right] + C$$



$$= \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{1}{12}\sin(2\theta) + C$$

$$= \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{1}{12}\cdot 2\sin\theta\cos\theta + C$$

$$= \boxed{\frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{2x^2} + C}$$

8) (6pts.) Evaluate the following integral. (*Hint: Make a substitution to change the exponential functions to polynomials. Then use partial fractions.*)

$$\int \frac{e^{2x}}{e^{2x}+3e^{x}+2}\,dx$$

$u = e^x$

$du = e^x dx$

$$\int \frac{e^x}{e^{2x}+3e^x+2}\, e^x dx = \int \frac{u}{u^2+3u+2}\, du$$

$$= \int \frac{u\, du}{(u+1)(u+2)} = \int \frac{-1}{u+1} + \frac{2}{u+2}\, du$$

$$= 2\ln|u+2| - \ln|u+1| + C$$

$$= \boxed{2\ln(e^x+2) - \ln(e^x+1) + C}$$

$$= \boxed{\ln\left(\frac{(e^x+2)^2}{e^x+1}\right) + C}$$

9) (3pts.) (a) Evaluate the integral $\displaystyle\int_1^{\infty} \frac{\ln x}{x}\, dx$

(3pts.) (b) Evaluate the integral $\displaystyle\int_e^{\infty} \frac{1}{x(\ln x)^3}\, dx$

a) $\displaystyle\int_1^{\infty} \frac{\ln(x)}{x}\, dx = \lim_{t\to\infty} \left.\frac{(\ln(x))^2}{2}\right|_1^t$

Use $u = \ln(x)$ ~~dx~~

$du = \frac{1}{x}\, dx$

$\displaystyle = \lim_{t\to\infty} \frac{(\ln(t))^2}{2} - \frac{\ln(1)^2}{2}$

$= \infty - 0$

$= \boxed{\infty \Rightarrow \text{Divergent}}$

$u = \ln(x)$

$du = \frac{1}{x} dx$

b) $\displaystyle\int_e^{\infty} \frac{1}{x(\ln(x))^3}\, dx = \lim_{t\to\infty} \int_e^t \frac{1}{x(\ln(x))^3} dx = \lim_{t\to\infty} \int_1^{\ln(t)} u^{-3} du$

$\displaystyle = \lim_{t\to\infty} \left.\left(-\frac{1}{2u^2}\right)\right|_1^{\ln t}$

$\displaystyle = \lim_{t\to\infty} \left(-\frac{1}{2(\ln(t))^2} + \frac{1}{2}\right) = 0 + \frac{1}{2} = \boxed{\frac{1}{2}}$

$\boxed{\text{Convergent}}$

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END OF TEST